ON COUNTABLY GENERATED IDEALS OF C(X)

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CERTIFICATE

This is to certify that the dissertation, "ON COUNTABLY GENERATED IDEALS OF C(X)" being submitted by Mr Prabudh Ram Misra for the award of the degree of Doctor of Philosophy of the Indian Institute of Technology Kanpur, is a record of bonafide research work In my opinion this manuscript has achieved the standards required for the Ph D degree It has not been submitted to any other University or Institute for the award of a degree or diploma

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CONTENTS

CHAPTER 1

1 INTRODUCTION

Let $C(X)$ be the ring of real valued continuous functions on a completely regular Hausdorff space X Hewitt [11] proved that if X is a realcompact metric space, then $C(X)$ completely determines X in the sense that a ring isomorphism between $C(X)$ and $C(Y)$ implies that X and Y are homeomorphic, for any other completely regular Hausdorff space Y But as Gillman and Jerison [9] point out, for practical purposes, metric spaces are realcompact and so $C(X)$ determines X for most of the metric spaces While working on metrization problems, Chittenden [2] studied the class of topological spaces in which every point is a zero-set Anderson [1] proved a remarkable theorem that if X and Y are two completely regular spaces in which every point is a zero-set, then a ring isomorphism between $C(X)$ and $C(Y)$ implies the existence of a homeomorphism between X and Y

In the thesis we prove that if X is a realcompact completely regular space in which every point is a zero-set, then $C(X)$ completely determines X in the sense of Hewitt Also, a different proof of Anderson's theorem is given These theorems are proved in chapter 4 by using the results developed in chapter 3

Since the topological spaces in which every point is a zero-set are important for us, in chapter 2 we study some topological properties of such spaces We call such spaces weakly perfectly normal (in brief, wpn) Since every closed set is a zero-set in a

perfectly normal space, a T_1 perfectly normal space will be weakly perfectly normal

The characterization of a point zero-set in terms of $C(X)$, presented in chapter 2, has been generalized for an arbitrary zero-set and hence we get a satisfactory characterization of perfect normality in terms of $C(X)$ All these characterizations are given in terms of the existence of certain countably generated ideals in $C(X)$ These in turn provide motivation for a study of those ideals of all the continuous functions vanishing on a neighborhood of a closed set of X and ideals of all the continuous functions vanishing on a closed set of X which are countably generated Furthermore, we study the ideals O^A of all the functions f in $C(X)$ such that $\overline{Z(f)}^{\beta X}$ is a neighborhood of A in βX and M^A of all the functions f in $C(X)$ such that $\overline{Z(f)}^{\beta X}$ contains A, where A is closed in βX, which are countably generated We settle a question raised in [5] by proving that in a pseudocompact space, M^A countably generated implies A is open, for any closed set A of βX Also, we give an example to show that it is not so in a general space All these things are dealt in chapter 3

From the results of chapter 3 we observe that countably generated ideals play an important role in metric spaces Kohls in [14] points out the scarcity of countably generated ideals in $C(X)$ One of the nice results proved in chapter 5 is that in a first countable space, a prime ideal is countably generated only if it is maximal and hence

it corresponds to an isolated point We next turn our attention to the study of countably generated z-ideals in $C(X)$, and identify some of them when X is a metric space In this context, there is a conjecture by De Marco [4] about the structure of countably generated z-ideals of $C(X)$ for an arbitrary space X The conjecture was settled in the affirmative by De Marco himself for compact spaces We settle his conjecture for pseudocompact spaces by the techniques developed in chapter 4

2 Notations and elementary results

The rest of this chapter is devoted to relevant notations, definitions and background results which will be needed throughout the thesis Some of the results were not known in the specific form stated here

By a space X we will always mean a completely regular Hausdorff space, excepting in the first five sections of chapter 2 The standard reference for the elementary definitions and results is [9]

If A is a subset of a space X, then A^c denotes the complement of A in X Some more notations are as follows.

$\mathbb{R}$ = Set of all real numbers

$\mathbb{Z}$ = Set of all integers

$\mathbb{N}$ = Set of all natural numbers

W = Set of all ordinals less than first uncountable ordinal

If $f \in C(X)$ then $Z_X(f) = \{x \in X \quad f(x) = 0\}$ If there is no chance of confusion, we write $Z(f)$ instead of $Z_X(f)$

<u>Proposition</u> 2 1 Every zero-set is a G_δ A closed G_δ is a zero-set if it is completely separated from any other disjoint closed set

The proof of the first statement is obvious and the proof of the second can be given by using the technique developed in 1 14 of [9] A simpler proof of it will be produced in 5th chapter

It is clear that $Z(f-|f|) = \{x \in X \quad f(x) \geq 0\}$ and $Z(f + |f|) = \{x \quad f(x) \leq 0\}$ We have the following results

<u>Proposition</u> 2 2 (a) Let $f \in C(X)$ There exists a function $g \in C(X)$ such that $Z(g) = f^{-1}[0,r]$ for some $r \in R$

(b) $f, g \in C(X)$ with $Z(g) \subset Z(f)$ Let b be a bounded continuous function on $X-Z(g)$ Define a function h on X as follows

$$h(x) = f(x) \quad b(x) \quad \text{for} \quad x \notin Z(g)$$

$$h(x) = 0 \qquad \text{for } x \in Z(g)$$

Then h is continuous on X

Proof (a) $g = (f - |f|)^2 + ((f - r) + |f - r|)^2$ serves the purpose

(b) Clearly h is continuous on $X-Z(g)$ Since $X-Z(g)$ is open in X, inverse image of any open set, not containing zero, in R, under h will be open in X as h is continuous on $X-Z(g)$ It remains to prove that inverse image of any open set of R, containing zero, is also open in X It is sufficient to prove that $h^{-1}(-\varepsilon, \varepsilon)$ is open for some $\varepsilon > 0$ As is argued above, it is clear that every point of $h^{-1}(t)$ is an interior point of $h^{-1}(-\varepsilon, \varepsilon)$ for any $t \neq 0$ and $t \in (-\varepsilon, \varepsilon)$ Next, let

$|b| < M$ Also, $|f(x)| < \frac{\varepsilon}{M} \Rightarrow -M|f(x)| > -\varepsilon \Rightarrow$ $b(x)\ |f(x)| > -\varepsilon$, and similarly $|f(x)| < \frac{\varepsilon}{M} \Rightarrow b(x)\ |f(x)| < \varepsilon$, we have

$$|f(x)| < \frac{\varepsilon}{M} \Rightarrow -\varepsilon < b(x)\ |f(x)| < \varepsilon$$

$$\Rightarrow -\varepsilon < b(x)\ \ f(x)\ < \varepsilon$$

$$\{x \ \ |f(x)| < \frac{\varepsilon}{M}\} \subseteq \{x \ \ -\varepsilon < h(x) < \varepsilon\}$$

But $Z(f) \subseteq \{x \ \ |f(x)| < \frac{\varepsilon}{M}\}$ which is open Thus every point of $Z(f)$ is an interior point of $h^{-1}(-\varepsilon, \varepsilon)$ We can write

$$h^{-1}(-\varepsilon, \varepsilon) = [(h^{-1}(-\varepsilon, \varepsilon) \cap (X - Z(g))] \cup Z(g)$$

Since $h^{-1}(-\varepsilon, \varepsilon) \cap (X-Z(g))$ is open in X and every point of $Z(g)$ is an interior point of $h^{-1}(-\varepsilon, \varepsilon)$, $h^{-1}(-\varepsilon, \varepsilon)$ is open in X #

<u>Proposition</u> 2 3 Let $f, g \in C(X)$

(a) If $Z(f)$ is a neighborhood of $Z(g)$, then f is a multiple of g

(b) There exists a positive unit u of $C(X)$ such that $(-1 \vee f) \wedge 1 = uf$

Proof (a) Let $h(x) = \frac{f(x)}{g(x)}$ for $x \notin \text{Int } Z(f)$, and $h(x) = 0$ for $x \in Z(f)$ Clearly h is a well defined function and when restricted to $X - \text{Int } Z(f)$ or $Z(f)$, it is continuous Thus h is continuous on whole of X It is now obvious that $h\,g = f$.

(b) Define a function $u \ \ X \to R$ such that $u(x) = 1$ for $x \in f^{-1}[-1,1]$, and $u(x) = \frac{1}{|f(x)|}$ for $x \in f^{-1}((-\infty,-1] \cup [1,\infty))$ Clearly u is well defined and is continuous on each of the closed sets $f^{-1}[-1,1]$ and $f^{-1}((-\infty,-1] \cup [1,\infty))$ Since $Z(u) = \emptyset$, u is invertible It is now plain that $(-1 \vee f) \wedge 1 = u\,f$ #

We give some definitions

<u>Definitions</u> 2 1 (a) By $(\ \ , f_\alpha, \ \ )$, $\alpha \in \Lambda$, where Λ is some indexing set, we mean the smallest ideal I containing f_α for each $\alpha \in \Lambda$ and say that the ideal I is generated by $\{f_\alpha \ \ \alpha \in \Lambda\}$

(b) $Z(I) = \{Z(f) \quad f \in I\}$

(c) $\bigcap Z(I) = \bigcap_{f \in I} Z(f)$

The proof of following theorem can be found in [9]

Theorem 2 4 Every z-ideal in $C(X)$ is an intersection of prime ideals

Definitions 2 2

(a) $O_A = \{f \in C(X) \quad A \subseteq \text{Int} \quad Z(f)\}$

(b) $M_A = \{f \in C(X) \quad A \subseteq Z(f)\}$, where A is a subset of X

It is clear that $O_A \subseteq M_A$ Also, it is very simple to observe that O_A and M_A are z-ideals, and that $O_A = \bigcap_{a \in A} O_a$ and $M_A = \bigcap_{a \in A} M_a$ We will prove the following result

Theorem 2 5 z-ideal is an algebraic property

Proof Let I be a z-ideal Given $f \in C(X)$, let there exist a $g \in I$ such that f belongs to every maximal ideal containing g Thus $p \in Z(g) \Longrightarrow g \in M_p \Longrightarrow f \in M_p \Longrightarrow p \in Z(f)$ We have $Z(g) \subseteq Z(f)$ Since I is a z-ideal, $f \in I$ Next, let I be some ideal in $C(X)$ and $g \in I$ Let there be an $f \in C(X)$ such that $Z(f) \supseteq Z(g)$ Since a maximal ideal is a z-ideal, f belongs to every maximal ideal containing g Hence if I has the property that, for given $f \in C(X)$, if there exists $g \in I$ such that f belongs to every maximal ideal containing g, then $f \in I$, then we observe that I is a z-ideal

We proved the following

I is a z-ideal if and only if given $f \in C(X)$, if there exists $g \in I$ such that f belongs to every maximal ideal containing g, then $f \in I$

From this characterization of z-ideals it is clear that if $C(X)$ is isomorphic to $C(Y)$, then image of a z-ideal will be a z-ideal #

We will not discuss the theory of Stone-Cech compactification and realcompactification for which we refer to [9] The following result is proved in [9]

Proposition 2 6 X is open in βX if and only if X is locally compact where βX is the Stone-Cech compactification of X

As an immediate corollary of this proposition, we get that $\beta R - R$ is closed in βR, and hence $\beta R - R$ is compact

Remark 2 1 It is simple to observe that if V is an open set in βX then $\overline{V \cap X}^{\beta X} = \overline{V}^{\beta X}$

Some more definitions are in order

Definitions 2 2

(a) $O^A = \{f \in C(X) \; A \subseteq \mathrm{Int}_{\beta X} \overline{Z(f)}^{\beta X}\}$, where A is a subset of X

(b) $M^A = \{f \in C(X) \; A \subseteq \overline{Z(f)}^{\beta X}\}$, where A is a subset of X

(c) $O_A^{\beta X} = \{f \in C(\beta X) \; A \subseteq \mathrm{Int}_{\beta X} Z_{\beta X}(f)\}$, where A is a subset of βX

(d) $M_A^{\beta X} = \{f \in C(\beta X) \; A \subseteq Z_{\beta X}(f)\}$, where A is a subset of βX

(e) $\phi(I) = \{p \in X \; M_p \supseteq I\} = \bigcap_{f \in I} Z(f) = \bigcap Z(I)$, for any ideal I in $C(X)$

(f) $\theta(I) = \{p \in \beta X \; M^p \supseteq I\} = \bigcap_{f \in I} \overline{Z(f)}^{\beta X}$, for any ideal I in $C(X)$

The following proposition is a direct consequence of above definitions

Proposition 2 7

(a) O^A and M^A are z-ideals

(b) $O^A = \bigcap_{a\varepsilon A} O^a$ and $M^A = \bigcap_{a\varepsilon A} M^a$

(c) $\phi(I) = X \cap \theta(I)$

(d) $O^{\theta(I)} \subseteq I$

Remark I need not contain $O_{\phi(I)}$

For, let $I = M^p \cap M_q$, then $\phi(I) = \{q\}$ and $O_{\phi(I)} = O_q$ Clearly I does not contain O_q as it is not contained in a unique maximal ideal

We here use some results of chapter 7 of [9].

We will require the following results in our study

Proposition 2 8 (a) If X is dense in T, then the family of all sets $\bar{Z}^T$, for $Z \varepsilon Z(X)$, is a base for the closed sets in T

(b) Every zero-set in βX is a countable intersection of sets of the form $\bar{Z}^{\beta X}$, for $Z \varepsilon Z(X)$, the collection of all zero-sets in X

(c) X is pseudocompact if and only if every nonempty zero-set in βX meets X

(d) X is pseudocompact if and only if $\beta X = \nu X$, where νX is the realcompactification of X

Proof (a) Let F be a closed set in T and $x \notin F$, where $x \varepsilon T$ Since T is completely regular, there exists a function $f \varepsilon C(T)$ such that $f(F) = 0$ and $f(x) = 1$ Consider a function $g \varepsilon C(T)$ such that $Z_T(g) = f^{-1}[0, \frac{1}{2}]$ (Proposition 1 2 2(a)) It is clear that $Z_T(g)$ is a neighborhood of F Let $\underline{g} = g/X$ From remark 1 2 1 it is clear that

$\overline{Z_X(g)}^T$ is a neighborhood of F since $Z_X(g) = Z_T(g) \cap X$. Further, since $\overline{Z_X(g)}^T \subseteq Z_T(g)$, $x \notin \overline{Z_X(g)}^T$ and the proposition is proved #

(b) Let $f \in C(\beta X)$. Choose $f_n \in C(\beta X)$ such that $Z_{\beta X}(f_n) = f^{-1}[0, \frac{1}{n}]$ (Proposition 1 2 2), for each $n \in N$. Since $Z_{\beta X}(f_n)$ is a neighborhood of $Z_{\beta X}(f)$, $Z_{\beta X}(f_n) \cap X \neq \emptyset$. Let $f_n' = f_n|X$. Clearly $Z_X(f_n') = Z_{\beta X}(f_n) \cap X$. Moreover, $\overline{Z_X(f_n')}^{\beta X}$ is a neighborhood of $Z(f)$ as is clear from remark 1 2 1. It is now clear that $\bigcap_{i=1}^{\infty} \overline{Z_X(f_n')}^{\beta X} = Z(f)$

(c) Let X be pseudocompact and $f \in C(\beta X)$ such that $Z_{\beta X}(f) \cap X = \emptyset$. If $\underline{f} = f|X$, then since $f^{-1}[0,\varepsilon)$ intersects X, $\underline{f}$ attains values in every neighborhood of zero. As $Z_{\beta X}(f) \cap X = \emptyset$, $Z_X(\underline{f}) = \emptyset$. Thus $\frac{1}{\underline{f}}$ exists and is unbounded on X, a contradiction. Conversely, let X be not pseudocompact. Let $f \in C(X)$ be unbounded. Thus, $|f|$ will be unbounded, and hence $\frac{1}{1 \vee |f|}$ will be bounded and continuous on X. Also, $\frac{1}{1 \vee |f|}$ will attain values in every neighborhood of zero and will never vanish in X. Let h be the continuous extension of $\frac{1}{1 \vee |f|}$ to βX. Clearly h assumes value zero at some points of βX. Hence $Z_{\beta X}(h) \neq \emptyset$ and $Z_{\beta X}(h) \cap X = \emptyset$, a contradiction

(d) Trivial #

<u>Proposition</u> 2 9 Let A be a closed set in βX. We have

$$A = \bigcap_{g \in O^A} \operatorname{Int} \overline{Z(g)}^{\beta X}$$

Proof Clearly $A \subseteq \bigcap_{g \in O^A} \operatorname{Int}_{\beta X} \overline{Z(g)}^{\beta X}$

Next, let $p \in \bigcap_{g \in O^A} \operatorname{Int}_{\beta X} \overline{Z(g)}^{\beta X}$ and $p \notin A$. Since A is closed in βX, by proposition (1 2 2) we get $\tilde{g}$ in $C(\beta X)$ such that $Z_{\beta X}(\tilde{g})$ is a

neighborhood of A and $p \notin Z_{\beta X}(\tilde{g})$, and hence, we get a function $g \in C(X)$ with $g \in O^A$ and $p \notin \overline{Z(g)}^{\beta X}$ Thus $p \notin \mathrm{Int}_{\beta X}\, \overline{Z(g)}^{\beta X}$ for one $g \in O^A$ This is a contradiction to $p \in \bigcap_{g \in O^A} \mathrm{Int}_{\beta X}\, \overline{Z(g)}^{\beta X}$ #

The proof of the following theorem can be found in [9]

<u>Theorem</u> 2 10 In any space, the union of a compact set with a realcompact set is realcompact

<u>Proposition</u> 2 11 In R, Z does not have a countable base of neighborhoods

Proof Let $\{U_1, U_2, \ \}$ be a decreasing sequence of neighborhoods of Z in R Let U_{p_1} be the first element of the sequence such that $U_1 \supsetneq U_{p_1}$ and the open interval in U_{p_1} containing 1 is strictly contained in the open interval in U_1 containing 1 Let us call the later property by "$U_1 \supsetneq U_{p_1}$ at 1" Now, get the first U_{p_2} in the sequence such that $U_{p_1} \supsetneq U_{p_2}$ at 2 Thus, continuing like this, we will get a subsequence $\{U_1, U_{p_1}, U_{p_2}, \ \}$ of the original sequence of neighborhoods of Z

Choose $q_1 \in U_1$ such that $q_1 \notin U_{p_1}$ and $q_1 \in$ the open interval in U_1 containing 1 Next, choose $q_2 \in U_{p_1}$ such that $q_2 \notin U_{p_2}$ and $q_2 \in$ the open interval in U_{p_1} containing 2 This way we will get a closed sequence $\{q_1, q_2, \ \}$ such that no q_i is an integer

Clearly no U_{p_i} is contained in the open set $R - \{q_1, q_2, \cdot \ \}$ containing Z Thus no U_i will be contained in this neighborhood of Z #

<u>Proposition</u> 2 12 A free ultrafilter cannot have a countable base

Proof If $\{U_n\}_{n\in N}$ is a countable base for a free filter F, then we can inductively pick two points from each U_n so as to obtain two disjoint sets, each of which meets every U_n, a contradiction to the fact that F is a filter

Another proof of this proposition will be given in example(2 5 3) in the next chapter

By example (2 5 3) we mean third example in section 5 of 2nd chapter Same rule will be followed for theorems, propositions, corollaries and remarks The theorems, propositions and corollaries in a section are numbered together

In the last we prove

<u>Proposition</u> 2 13 (a) X is a P-space if and only if $M^p = O^p$ for all $p \in \beta X$

(b) A pseudocompact P-space is finite

Proof (a) Since a space X is a P-space if and only if $M_p = O_p$ for all $p \in X$ and since $M^p = M_p$ for all $p \in X$, the converse is clear If X is a P-space, then to prove that $M^p = O^p$ for all $p \in \beta X$, it is sufficient to show that $M^p - O^p$ for each p in βX Let $f \in M^p$ Thus $p \in \overline{Z(f)}^{\beta X}$ Since $Z(f)$ is a G_δ in X, it is open in X as X is a P-space $X-Z(f)$, being open and closed both is zero-set in X By the characteristic property of βX, $\overline{(X-Z(f)) \cap Z(f)}^{\beta X} = \emptyset = \overline{(X - Z(f))}^{\beta X} \cap \overline{Z(f)}^{\beta X}$ Since $\overline{(X - Z(f))}^{\beta X} \cup \overline{Z(f)}^{\beta X} = \overline{(X-Z(f)) \cup Z(f)}^{\beta X}$ $= \beta X, \overline{Z(f)}^{\beta X}$ is open as well as closed in βX We get that $\mathrm{Int}_{\beta X}\, \overline{Z(f)}^{\beta X} = \overline{Z(f)}^{\beta X}$, and hence $f \in O^p$

(b) It is plain that X is a P-space if and only if βX is a P-space But a compact P-space is discrete and hence it is finite We get that βX is finite, which in turn gives that X is finite #

Let F be a subset of a space X Then, by X/F we mean the quotient space of X obtained by identifying F with a point Thus if τ $X \to X/F$ is the quotient map, a set $G \subseteq X/F$ is open in X/F if and only if $\tau^{-1}(G)$ is open in X

CHAPTER 2

WEAK PERFECT NORMALITY

This chapter is mostly devoted to the study of weakly perfectly normal spaces We prove some topological properties of such spaces and discuss their relationships with the separation axioms The spaces considered in first five sections are not completely regular unless otherwise mentioned From sixth section onwards all the spaces considered will be completely regular and hence weakly perfectly normal spaces coincide with G_δ-spaces defined by Anderson [1] after fifth section In the last section we present some results for point zero-sets proved by Kohls [13] which will be needed in the last chapter

§1 Definition and examples

Definition 1 1 A space X is weakly perfectly normal (wpn) if every point of X is a zero-set

Example 1 1 Every metric space is wpn In fact, every metric space is T_1, perfectly normal, and we prove the following

Proposition 1 1 A space X is perfectly normal iff every closed set in X is a zero-set

Proof X is perfectly normal $\Longleftrightarrow$ X is normal and every closed set is a G_δ in X $\Longleftrightarrow$ every closed set is a zero-set by proposition (1 2 1)

Since a space is T_1 iff every point in the space is closed, a T_1 perfectly normal space is weakly perfectly normal Since non-T_1

perfectly normal spaces are not of much interest, we give the name weakly perfectly normal to the space in which every point is a zero set as this property is implied by T_1, perfect normality

Example 1 2 We give an example of a wpn space which is not regular

Let S denote the subspace of $R \times R$ obtained by deleting (0,0) and all points $(\frac{1}{n}, y)$ with $y \neq 0$ and $n \in N$ Define $\pi(x,y) = x$ for all $(x,y) \in S$, then π is a continuous mapping of S onto R Let E denote the quotient space of S associated with the mapping π Thus, a set $A \subset E$ is open in E if and only if $\pi^{-1}(A)$ is open in S

(a) E is wpn It is sufficient to prove that $C(R) \subset C(E)$ But this fact is obvious as E has stronger topology than R (See also corollary 2 2 3)

(b) E is not regular Since $\pi^{-1}(R - \{\frac{1}{n}\}_{n \in N}) = S - \{(\frac{1}{n},0)\}_{n \in N}$, which is clearly open in S, we get that $R - \{\frac{1}{n}\}_{n \in N}$ is open in E Thus $\{\frac{1}{n}\}_{n \in N}$ is closed in E Since the point 0 in E has the usual neighborhood system in E, we can not separate 0 and $\{\frac{1}{n}\}_{n \in N}$ by disjoint open sets containing them

Proposition 1 2 A wpn space is functionally Hausdorff Thus it is Hausdorff and T_1

Proof Let $x, y \in X$ where X is wpn There exist $f_x, f_y \in C(X)$ such that $Z(f_x) = \{x\}$ and $Z(f_y) = \{y\}$ Consider the function

$f = \dfrac{|f_x|}{|f_x| + |f_y|}$ Since $Z(|f_x| + |f_y|) = \emptyset$, f is well defined continuous function Clearly $f(x) = 0$ and $f(y) = 1$ Thus X is functionally Hausdorff

Example 1 3 If X is non-compact completely regular, Hausdorff, we know that no point of $\beta X - X$ is a G_δ Since βX is compact Hausdorff, it is normal and hence functionally Hausdorff Thus if X is non-compact, CR, T_2 space then βX is an example of a functionally Hausdorff space which is not wpn as every zero-set must be G_δ In particular, since $\beta R - R$ is compact Hausdorff (Proposition 1 2 6) $\beta R - R$ is an example of a functionally Hausdorff space which is not wpn In fact no point of $\beta R - R$ is a G_δ Later we shall give an example of a functionally Hausdorff space in which a G_δ point is not a zero-set

Remark 1 1 βX is wpn if and only if X is compact and wpn

Example 1 4 A countable wpn space need not be first countable

The space $\sum$ Let $\mathcal{U}$ be a free ultrafilter on N (e g , the collection of all the subsets of N whose complements are finite is contained in a free ultra-filter on N)

Let $\sum = N \cup \{\sigma\}$ (where $\sigma \notin N$)

Define a topology on $\sum$ as follows All points of N are isolated, and the neighborhoods of σ are the set $U \cup \{\sigma\}$ for $U \in \mathcal{U}$

It is clear that $\sum$ is Hausdorff Since complement of any subset of $\sum$ containing σ is a subset of N , such a subset will be

open Thus, any subset of σ containing σ is closed and if it does not contain σ, it will be open Let F be a closed set in $\sum$ If $\sigma \notin F$, then F will be open and hence will be a zero-set If $\sigma \in F$, then $\sum - F \subseteq N$, and define f $\sum \to P$ such that $f(n) = \frac{1}{n}$ if $n \in \sum - F$ and $f(p) = 0$ for each $p \in F$ Clearly f is continuous and $Z(f) = F$ Thus every closed subset of $\sum$ is a zero set Since points are closed in $\sum$, $\sum$ is wpn

Since a free ultrafilter cannot have a countable base (1 2 12) the point σ does not have a countable base of neighborhoods Hence $\sum$ is not metrizable

The next example of wpn space which is not first countable is very important for us We will use this example at several places

<u>Example</u> 1 5 We have observed in chapter 1 (1 2 13) that Z does not have a countable base of neighborhoods in R If we identify Z with the point zero in R and denote the resulting space with quotient topology by R/Z, then clearly the point 0 in R/Z, determined by Z will not have a countable base of neighborhoods But since Z is a zero set in R, the point 0 is a zero-set in R/Z Any other point of R/Z is also a zero-set as is clearly seen Thus R/Z is a wpn space which is not first countable

§2 Various images of a wpn space

<u>Proposition</u> 2 1 Let X be a wpn space and f $X \to Y$ be a one-one open map Then $f(X)$ is wpn

Proof Let $g_x : X \to R$ be a continuous function such that $Z(g_x) = \{x\}$, for each $x \in X$. Define $h_{f(x)} : f(X) \to R$ such that $h_{f(x)}(y) = g_x(f^{-1}(y))$. Since f is open, clearly $h_{f(x)}$ is continuous. Next, $Z(h_{f(x)}) = Z(g_x \circ f^{-1}) = \{f(x)\}$. The proposition is clear. #

Since a one-one closed map is open, we have the following

Corollary 2.2 One-one closed image of a wpn space is wpn. #

Corollary 2.3 Stronger topology than a wpn topology is wpn. #

Also, since the inverse of a one-one onto continuous map is open, we have

Corollary 2.4 Let $f : X \to Y$ be one-one continuous and Y be wpn. Then X is wpn. #

The proof of this corollary is clear from the following

Corollary 2.5 Weak perfect normality is hereditary.

Proof Restriction of a continuous function to a subspace is continuous. #

The quotient $R/\sim$ of R by the relation $x \sim y$ iff $x-y$ is rational with the quotient topology is indiscrete. Since the projection $P : R \to R/\sim$ is open, we observe that open continuous image of a metric space need not be wpn. The example R/Z of section 1 shows that even if the quotient of a metric space is wpn, it need not be first countable. Since R/Z is quotient of the metric space R, it is sequential, i.e., a subset U of R/Z is open if and only if each sequence in R/Z converging to a point

in U is eventually in U (Franklin [7]) Thus we get an example of a sequential wpn space which is not first countable A little stronger counter example can also be obtained A Frechet space (i e , closure of any subset A of such a space is the set of limits of sequences in A) is a sequential space and the space $\sum$ of section 1 is an example of a Frechet space which is wpn but not first countable We need only to prove that $\sum$ is Frechet

Let A be a set in $\sum$ If $\sigma \in A$, then A is closed Let $\sigma \notin A$ Then since $\mathcal{U}$ is an ultrafilter, either $A \in \mathcal{U}$ or $X-A \in \mathcal{U}$ If $X-A \in \mathcal{U}$, $\sigma \notin \bar{A}$ and $A = \bar{A}$ If $A \in \mathcal{U}$, then A is not closed and $\bar{A} = A \cup \{\sigma\}$ In this case, since A is infinite, it is an infinite sequence converging to σ Thus $\bar{A}$ = set of limits of sequences in A, and hence $\sum$ is Frechet

We have proved the following

<u>Proposition</u> 2 6 A wpn Frechet spae need not be first countable #

Since a Hausdorff space is a k-space if and only if it is a quotient space of a locally compact space, the space R/Z reflects the following

<u>Proposition</u> 2 7 A wpn k-space need not be first countable #

Continuing the discussion about the images of a wpn space, we now give an example to show that a closed continuous image of a wpn space need not be wpn

<u>Example</u> 2 1 It is easily seen that W is wpn Identifying all the limit ordinals (which form a closed set in W) in W and consider

the quotient topology induced by W on this set. Let W_e denote this space. Clearly this quotient map is closed. The point in W_e determined by the set of all limit ordinals cannot be a zero-set because otherwise, the set of all limit ordinals will be a zero-set in W, which is a contradiction to the fact that this set is not a G_δ in W(any continuous function on W is constant on a tail)

§3 Disjoint sum and product

Proposition 3.1 Disjoint sum of wpn spaces is wpn

Proof. Let X_α be a wpn space and let $X^*_\alpha = \{(x,\alpha) | x \varepsilon X_\alpha\}$ be a homeomorphic copy of X_α, for each $\alpha \varepsilon A$. The disjoint sum of the spaces $X_\alpha(\alpha \varepsilon A)$ is the space $\sum_{\alpha \varepsilon A} X_\alpha$ with the underlying set $\bigcup_{\alpha \varepsilon A} X^*_\alpha$ and the topology: $U \subset \sum_{\alpha \varepsilon A} X_\alpha$ is open if and only if $U \cap X^*_\alpha$ is open for each $\alpha \varepsilon A$. Then clearly the function $\sum_{\alpha \varepsilon A} f_\alpha$ defined on $\sum_{\alpha \varepsilon A} X_\alpha$ such that $\sum_{\alpha \varepsilon A} f_c | X^*_\alpha = f_\alpha$ is continuous if and only if each f_α is continuous.

Let $(x,\alpha) \varepsilon \sum_{\alpha \varepsilon A} X_\alpha$. We have to show that (x,α) is a zero-set. Since $(x,\alpha) \varepsilon X^*_\alpha$, which is homeomorphic to X_α, (x,α) is a zero-set in X^*_α. Let $f_{\alpha x} \varepsilon C(X^*_\alpha)$ such that $Z(f_{\alpha x}) = \{(x,\alpha)\}$. Define $\sum_{\alpha \varepsilon A} f_\alpha$ on $\sum_{\alpha \varepsilon A} X$ such that $\sum f_\alpha / X^*_\beta = 1$ on X_β if $\beta \neq \alpha$, and $\sum_{\alpha \varepsilon A} f_\alpha / X^*_\alpha = f_{\alpha x}$. Then $\sum_{\alpha \varepsilon A} f_\alpha$ is continuous and $Z(\sum_{\alpha \varepsilon A} f_\alpha) = \{(x,\alpha)\}$

Proposition 3.2 Let $\{X_\alpha\}_{\alpha \varepsilon A}$ be a family of wpn spaces. Then $\prod_{\alpha \varepsilon A} X_\alpha$ is wpn if and only if each X_α is wpn and, excepting for countably many $\alpha \varepsilon A$, each X_α is singleton

Proof Let $\prod_{\alpha\in A} X_\alpha$ be wpn Hereditary property of weak perfect normality implies the weak perfect normality of X_α, for each $\alpha \in A$ The next fact follows by observing that the elements in a base of topology on $\prod_{\alpha\in A} X_\alpha$ have proper subsets of X_α as fibers only for finitely many indices and hence any G_δ-set in $\prod_{\alpha\in A} X_\alpha$ will have proper subsets as fibers for at the most countably many indices

Conversely, let each X_α be singleton, for $\alpha \notin B$, where B is a countable subset of A Without loss of generality we can assume $B = N$ Let $f_n \in C(X_n)$ Define $\tilde{f}_n$ to be the function $|f_n| \circ P_n$, where P_n is the nth projection from $\prod_{\alpha\in A} X_\alpha$ to X_n, for each $n \in A$ Clearly $\tilde{f}_n \in C(\prod_{\alpha\in A} X_\alpha)$ whenever $f_n \in C(X_n)$, for each $n \in A$ To show that $(x_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha$ is a zero-set, choose $f_n \in C(X_n)$ such that $Z(f_n) = \{x_n\} \subset X_n$, for each $n \in N$ Now $\sum_{n\in N} 2^{-n} \tilde{f}_n$ is a uniformly convergent series of continuous functions $(\in C(\prod_{\alpha\in A} X_\alpha))$, hence is continuous on $\prod_{\alpha\in A} X_\alpha$ Also $Z(\sum_{n\in B} 2^{-n} \tilde{f}_n) = \{(x_\alpha)_{\alpha\in A}\}$ #

§4 Covering properties

<u>Proposition</u> 4 1 A first countable completely regular Hausdorff space is weakly perfectly normal

Proof Let X be a first countable completely regular Hausdorff space and $p \in X$ Let $\{U_i\}_{i\in N}$ be a base of neighborhoods for the point p Since the space is Hausdorff, $p = \bigcap_{i=1}^{\infty} U_i$ Thus p is a G_δ in X, which is a completely regular space By proposition 2 5 1 we get that p is a zero-set #

Question 1 Can we replace complete regularity by functionally Hausdorffness in the above proposition ?

The converse of above proposition is not true as is shown by examples 2 1 4 and 2 1 5 In the following proposition we discuss sufficient conditions on a wpn space to imply first countability In fact, we shall study covering properties of a wpn space A similar proposition for pseudo-compact spaces will be proved in the fourth chapter (Corollary (4 1 3))

<u>Proposition</u> 4 2 In a countably compact space X every closed regular G_δ(hence a zero-set) has a countable base of neighborhoods

Proof Let A be a closed regular G_δ set in X So $A = \bigcap_{i=1}^{\infty} U_i = \bigcap_{i=1}^{\infty} \overline{U}_i$ where U_i is open in X, for every $i \in N$

Claim $\{W_p = \bigcap_{j \leq p} U_j\}_{p \in N}$ is a base of neighborhoods for A

For, let V be an open set containing the set A Then $\{V, \overline{W}_p^C\}_{p \in N}$ forms an open countable covering of X Since X is countably compact, this covering has a finite subcover, say $\{V, \overline{W}_{p_1}^C, \overline{W}_{p_2}^C, \ldots, \overline{W}_{p_n}^C\}$ Let $p_i = \max\{p_1, \ldots, p_n\}$ Then clearly, since $W_{p_i+1} \subset W_{p_i}$, $\overline{W}_{p_j}^C$ does not intersect W_{p_i+1} for any j with $1 \leq j \leq n$ Thus $W_{p_i+1} \subset V$ #

<u>Corollary</u> 4 3 In a countably compact normal space every closed G_δ has a countable base of neighborhoods #

<u>Corollary</u> 4 4 In a compact Hausdorff space every closed G_δ has a countable base of neighborhoods #

Corollary 4 5 A countably compact wpn space is first countable Thus a compact wpn space is first countable #

Corollary 4 6 In a countably compact regular space every G_δ-point has a countable base of nbhds #

Proposition 4 7 A locally compact wpn space X is first countable

Proof Let $x \in X$ be such that $x \in U \subset C$ where C is compact and U is open in X Since wpn is hereditary, C is also wpn Thus by corollary 2 4 4 of the above proposition, C is first countable Let $\{V_i\}_{i \in N}$ be a countable base for the point x in C Since C is a subspace of X, $V_i = W_i \cap C \ \forall i \in N$, where W_i is some open set in X Since $U \subset C$, $U_i = W_i \cap U \subset W_i \cap C$ Thus, since U is open in X, U_i will be open in X and $U_i \subset V_i \ \forall i \in N$ Clearly $\{U_i\}_{i \in N}$ is a countable base of neighborhoods for the point x in X #

We sum up the above propositions and corollaries to get

Theorem 4 8 A wpn space with any of the following properties is first countable

(1) Countable compactness

(2) Sequential compactness

(3) Compactness

(4) Local compactness

(5) Completely regular and pseudocompactness

(6) M-compactness

The proof of (5) will be given in the fourth chapter and (6) follows from (1) Also, it is interesting to note that a paracompact

wpn space need not be first countable In fact P/Z is wpn, paracompact but is not first countable as was pointed out in Example 2 1 5

§5 G_δ-spaces

F W Anderson [1] studied the class of topological spaces in which every point is a G_δ He calls such spaces G_δ spaces The purpose of this section is to compare the G_δ spaces with wpn spaces In view of proposition (1 2 1) it is clear that a wpn space is a G_δ-space The converse is true in a completely regular space This is proved in the following proposition It is easy to observe that a G_δ space is always T_1 but need not be T_2

<u>Example</u> 5 1 Let X be any countable set with cofinite topology Clearly X is a non-Hausdorff compact G_δ-space which is connected also Since X is countable and connected, the only real valued continuous functions on X are constants

Below we shall give an example (viz example 2 5 2) of a functionally Hausdorff G_δ space which is not wpn

<u>Proposition</u> 5 1 A completely regular G_δ-space X is wpn

Proof Let $p \in X$ Then $p = \bigcap_{i=1}^{\infty} U_i$ where each U_i is open in X Since $p \notin X - U_i$ and X is completely regular, there exists a zero-set $Z(f_i)$ such that $p \in Z(f_i) \subset U_i$ Thus $p = \bigcap_{i=1}^{\infty} Z(f_i)$ and hence a zero-set, being a countable intersection of zero-sets #

The space Γ/Q in the following example is functionally Hausdorff G_δ-space which is not wpn

Example 5 2 The space Γ Let Γ denote the subset $\{(x,y) \mid y \geq 0\}$ of $R \times R$, with the following topology The neighborhood system for any (x,y), $y > 0$, is the same as in $R \times R$ with the usual product topology For each $x \in R$ the point $(x,0)$ has neighborhoods $V_r(x,0) = \{(x,0)\} \cup \{(u,v) \in \Gamma \quad (u-x)^2 + (v-r)^2 < r^2\}$, for $r > 0$, together with the neighborhoods coming from the usual product topology on $R \times R$

It is clear that Γ is first countable The subspace $D = \{(x,0) \quad x \in R\}$ of Γ is discrete Consider $f \quad \Gamma \to R_+$ such that $f(x,t) = t$ Clearly f is continuous and $Z(f) = D$ Thus D is a zero-set Next we prove that Γ is completely regular Let F be a closed set in Γ and $x \notin F$ Since $R \times R$ with the product topology is completely regular, only case to be considered is that $x \in D$ and $F = \Gamma - V_r(x)$

Let B_r be the boundary of $V_r(x)$ Define a function f such that $f(x) = 0$ and $f(B_r) = 1$ Let $y \in B_r$ Define f linearly on the segment determined by the points x and y Also define $f(X-V_r(x)) = 1$ Consider the neighborhood $[0,\varepsilon)$ of 0 in R_+ Clearly $f^{-1}\,[0,\varepsilon) = V_{r_1}(x)$ Similarly inverse images of other open sets of R_+ are also open and we get $f \in C(\Gamma)$, which separates x and F Hence Γ is completely regular

Let Q denote the set of rational numbers in the sub-space D Clearly Q is closed in Γ Let f be a positive continuous functions which vanishes on Q Due to continuity of f, it is clear that f

will go on decreasing on the line $\{(i,x) \quad x \in R\}$, $x \neq 0$, for any $i \in P$, and hence inparticular it goes on decreasing on any line above an irrational point of D We get that f should vanish on $(i,0)$ for each irrational number $i \in R$, Thus Q can not be a zero-set in Γ Clearly Γ/Q is a functionally Hausdorff space in which $\bar{Q}$, the point determined by Q, is a G_δ but not a zero-set

Since a G_δ set is open in a P-space, a G_δ space which is also a P-space is discrete But a wpn F-space need not be discrete

Example 5 3 The space Σ (Example 2 1 4) is an F-space

For, $n \in N$, $O_n = M_n$ is clear Thus each O_n is maximal We prove that O_σ is prime Let $fg \in O_\sigma$ $Z(fg) = Z(f) \cup Z(g)$ is a zero set neighborhood of σ, hence $(Z(f) - \{\sigma\}) \cup (Z(g) - \{\sigma\}) \in \mathcal{U}$ Hence either $Z(f) - \{\sigma\}$ or $Z(g) - \{\sigma\}$ belongs to $\mathcal{U}$, and we get that Σ is an F-space

Since it is easy to see that a first countable F-space is discrete (Cor 5 1 5), we get another proof of the fact that the point σ in the space Σ does not have a countable base of neighborhoods and hence that any free ultrafilter on N does not have a countable base Since N does not play any role in obtaining these observations, we can prove that a free ultrafilter on any set S cannot have a countable base by using the above technique Observe that the space constructed from S analogous to Σ will fail to be wpn unless S is countable

§6 Characterization of weak perfect normality in $C(X)$ and some known results

To carry out our study via $C(X)$, we always assume X to be completely regular because for any space X, there exists a completely regular space Y such that $C(X)$ and $C(Y)$ are isomorphic We shall call such a space Y, the complete regularization of X

Now onwards all the spaces considered will be completely regular Thus Anderson's G_δ-spaces and wpn spaces are one and the same hereafter

We state the following well known results [9]

<u>Proposition</u> 6 1 Let X be completely regular Then,

(1) $p \in X$ is a zero-set if and only if $O_p \subset I \subset M_p$ with I countably generated

(2) $p \in X$ has a countable base of neighborhoods if and only if O_p is countably generated

Proof Corollaries of Lemma 3 1 1 and proposition 3 2 1 respectively

<u>Remark</u> 6 1 In (1) above it is clear from the proof that, the term "countably generated" can be replaced by "principal" Since O_p finitely generated means p is isolated (Prop 3 2 6), in (2) the term "countably generated" can not be replaced by "finitely generated" or "principal"

The characterizations in the proposition are not algebraically invariant as is shown by the spaces W and W^* In fact, in the category

of real compact spaces, the above characterizations are algebraically invariant and we show more strongly that if X is real-compact wpn and if $C(X)$ is isomorphic to $C(Y)$, then not only that Y will be wpn but also Y will be homeomorphic to X This fact is proved in fourth chapter

The above proposition motivates us to ask whether we can get a similar characterization for any zero-set of X ? The answer is affirmative and we prove this in the next chapter

We next prove

Lemma 6 2 Let p be a non-isolated zero-set in X If $Z \in Z(C(X-\{p\}))$, then $\bar{Z}^X \in Z(C(X))$

Proof Clearly $\bar{Z}^X = Z$ or $Z \cup \{p\}$ Let $h \in C(X-\{p\})$ with $Z(h) = Z$ and $0 \leq h \leq 1$ on $X - \{p\}$

Case 1 $\bar{Z}^X = Z \cup \{p\}$ Since p is a zero-set in X, $\exists$ a non negative function $f_1 \in C(X)$ such that $Z(f_1) = \{p\}$ Define a function g on X by $g(x) = f_1(x)\, h(x)$ for $x \neq p$ and $g(p) = 0$ Then $g \in C(X)$ by proposition (1 2 2(b)), and also $Z(g) = Z \cup \{p\}$

Case 2 $\bar{Z}^X = Z$ By complete regularity, there exists a function $f_2 \in C(X)$ such that $f_2(Z) = 1$, $f_2(p) = 0$ and $0 \leq f_2 \leq 1$ on X Hence if f_1 is the function chose in case 1, we have $f_1 + f_2 \geq 0$ on X Thus if $f_3 = 1 \wedge (f_1+f_2)$, we have $f_3 = 1$ on Z, $Z(f_3) = \{p\}$ and $0 \leq f_3 \leq 1$ on X Finally, define a function k on X by $k(x) = f_3(x)\,(1-h(x))$ for $x \neq p$ and $k(p) = 0$ Then $k \in C(X)$, by proposition (1 2 2(b)) & $Z(1-k)=Z$ #

The above lemma enables us to define a mapping $r \; Z(C(X-\{p\})) \to Z(C(X))$ by $r(Z) = \overline{Z}^X$ where $Z \in Z(C(X-\{p\}))$

The following properties of r are obvious

1 If F is a zero-set in X, then $r(F-\{p\})$ is either $F-\{p\}$ or F

2 If F is a zero-set of $X-\{p\}$, then $r(F) - \{p\} = F$

We also have the following

3 If $\mathcal{P}$ is a prime z-filter on X contained properly in $Z(M_p)$, and $Z \in \mathcal{P}$, then $r(Z-\{p\}) = Z$

Proof $r(Z-\{p\}) = Z-\{p\}$ or Z Also $Z(O_p) \subseteq \mathcal{P} \subsetneq Z(M_p) \Longrightarrow$ p is non-isolated in Z Thus $p \in \overline{Z-\{p\}}^X \subseteq Z$ #

Let $i \; X - \{p\} \to X$ be the inclusion Let ϕ denote the extension of i to the largest sub-space X_1 of $\beta(X-\{p\})$ to which it is extendable as a continuous function into X

Claim ϕ is closed

Let j be the inclusion function from $X-\{p\}$ into βX Since p is non-isolated, $X-\{p\}$ is not compact Consider the extension $\hat{j}$ of j to $\beta(X-\{p\})$ It is easily seen that $\hat{j}^{-1}(X) = X_1$, and hence $\phi = \hat{j}|X_1$ Let A be closed in X_1 & B closed in $\beta(X-\{p\})$ such that $B \cap X_1 = A$ Since B is closed in $\beta(X-\{p\})$ it is compact, hence $\hat{j}(B)$ is closed in βX

$$\text{Now } \hat{j}(B) = \hat{j}((B-A)\cup A) = \hat{j}(B-A)\cup\hat{j}(A)$$

$$\begin{aligned} \text{Hence } \hat{j}(B)\cap X &= (\hat{j}(B-A)\cup \hat{j}(A))\cap X \\ &= (\hat{j}(B-A)\cap X)\cup(\hat{j}(A)\cap X) \\ &= \emptyset\cup\hat{j}(A), \text{ because} \end{aligned}$$

$B-A \cap X_1 = \emptyset$ and $X_1 = \hat{j}^{-1}(X)$ Thus $\hat{j}(A)$ is closed in X But $\hat{j}|X_1 = \phi$ Hence ϕ is closed Also, $X_1 - (X-\{p\})$ cannot be empty as ϕ is closed, ϕ on $X-\{p\}$ is identity and p is non-isolated Thus $\phi^{-1}(\{p\}) \neq \emptyset$

The facts developed above make the proof of the following result due to Kohls [13] routine

Theorem (Kohls) 6 3 Let p be a non-isolated zero-zet in a space X, and let ϕ and r be defined as above Then,

(a) If $\mathcal{W}$ is a prime z-filter on $X-\{p\}$ converging to a point of $\phi^{-1}(\{p\})$, then $r(\mathcal{W})$ is a prime z-filter on X contained properly in $Z(M_p)$

(b) If $\mathcal{V}$ is a prime z-filter on X contained properly in $Z(M_p)$, then $r^{-1}(\mathcal{V}) = \{V-\{p\} \quad V \in \mathcal{V}\}$, and $r^{-1}(\mathcal{V})$ is a prime z-filter on $X-\{p\}$ converging to a point of $\phi^{-1}(\{p\})$

(c) The mapping r is one-to-one from the set of prime z-filters on $X-\{p\}$ converging to points of $\phi^{-1}(\{p\})$ onto the set of prime z-filters on X contained properly in $Z(M_p)$

(d) A prime z-filter $\mathcal{W}$ on $X-\{p\}$ converging to a point of $\phi^{-1}(\{p\})$ is a z-ultrafilter if and only if $r(\mathcal{W})$ is maximal in the class of (prime) z-filter on X contained properly in $Z(M_p)$

We use part (d) to prove the next theorem which will be used in chapter 5 to get some information about prime ideals in $C(X)$ for a first countable space X

Theorem 6 4 For a non-isolated G_δ-point p, the following are equivalent

1 In the class of prime z-ideals contained properly in M_p, there is exactly one maximal element Q

2 p is a point from $\beta(X-\{p\})$

3 $X - \{p\}$ is C* embedded in X

Proof 1) $\Longrightarrow$ 2) Part (d) of the last lemma implies that there exists only one z-ultrafilter on $X - \{p\}$ converging to a point of $\phi^{-1}(\{p\})$ Since $\phi^{-1}(\{p\}) \subset \beta(X-\{p\})-(X-\{p\})$, $\phi^{-1}(\{p\})$ is singleton Thus ϕ is one-to-one onto from X_1 to X Also ϕ is closed and continuous We have X_1 homeomorphic to X Thus we can consider that $X \subset \beta(X-\{p\})$ as $X_1 \subset \beta(X-\{p\})$ 2) $\Longrightarrow$ 1) We have $X \subset \beta(X-\{p\})$ Since X_1 is largest subspace of $\beta(X-\{p\})$ to which i can be continuously extended, $p \in X_1-(X-\{p\})$ Thus $i \; X - \{p\} \to X$ has extension $i_X \; X \to X$, the identity map on X Since a continuous function on βX has unique extension to X, the extension of i_X considered as the inclusion map from X to βX, to βX will be identity on βX Thus it is not possible to extend the function $i \; X - \{p\} \to X$ beyond X otherwise the range will no more restrict to X Thus $\phi^{-1}(\{p\})$ is singleton and again part (d) of last lemma gives us the result 2) $\Longleftrightarrow$ 3) trivial

§ Miscellaneous Remarks

1 Completely normal + wpn $\not\Longleftrightarrow$ perfectly normal (e g W)

2 Completely regular + wpn $\not\Longleftrightarrow$ normal (e g Γ)

3 Since the known examples of regular spaces which are not completely regular fail to be wpn, it will be interesting to know if a regular wpn space is completely regular Also, we do not have an example of a normal wpn space which is not completely normal

4 We know that a minimal Hausdorff space which is also Urysohn is always compact Thus a minimal Hausdorff wpn space is compact It would also be of interest to study minimal wpn spaces

CHAPTER 3

EXISTENCE OF COUNTABLY GENERATED IDEALS IN C(X)

The characterization of weak perfect normality and first countability show the existence of countably generated ideals in between O_p and M_p. Thus it is natural to study the conditions on a closed set A such that O_A is contained in a countably generated ideal or in particular O_A or M_A is countably generated. In the present chapter we identify these countably and finitely generated ideals and obtain very nice characterization for perfect normality. The countable and finite generation of O^A and M^A are also studied for A closed in βX. Some applications of the results obtained are given.

1. Perfect Normality and normality

<u>Lemma</u> 1.1. Let X be a completely regular space. A closed set F in X is a zero-set in X if and only if there exists a countably generated ideal between O_F and M_F.

Proof. Let $F = Z(f)$ for some $f \in C(X)$ and $g \in O_F$. Since $Z(g)$ is a neighborhood of $Z(f)$, proposition (1.2.3(a)) implies that g is a multiple of f and hence $g \in (f)$, the principal ideal generated by f. Next, $Z(f) = F \Longrightarrow f \in M_F$. We have $O_F \subseteq (f) \subsetneq M_F$.

Conversely, let F be closed with $O_F \subsetneq I \subseteq M_F$ where $I=(f_1, f_2, \ldots)$. It is easy to see that $\bigcap_{f\in I} Z(f) = \bigcap_{i\in N} Z(f_i)$. Thus $\bigcap Z(I) (= \bigcap_{f\in I} Z(f))$, being countable intersection of zero-sets, is a zero-set.

Claim $\bigcap Z(I) = \bigcap_{f \in O_F} Z(f) = \bigcap_{f \in M_F} Z(f) = F$

For, $\bigcap_{f \in O_F} Z(f) \supseteq \bigcap_{f \in I} Z(f) \supseteq \bigcap_{f \in M_F} Z(f) \supseteq F$

is clear Let $x \in \bigcap_{f \in O_F} Z(f)$ If $x \notin F$, then there exists $h \in C(X)$ such that $h(F) = 0$ and $h(x) = 1$ We get a $g \in C(X)$ with $Z(g) = h^{-1}[0,\frac{1}{2}]$ and $x \notin Z(g)$ from proposition (1 2 2(a)) This is a contradiction to $g \in O_F$ Thus the claim is true Since $\bigcap Z(I)$ is a zero-set, F is also a zero-set

Since a space X is perfectly normal if and only if every closed set in X is a zero-set (Proposition 2 1 1), we immediately have

<u>Theorem</u> 1 2 A space X is perfectly normal if and only if for each closed set F in X there exists a countably generated ideal between O_F and M_F

<u>Remark</u> 1 1 As is clear from the proof of the lemma, the term "countably generated" can be replaced by "principal" in the lemma and the theorem It will be observed later that the principal ideal constructed in the above proof can not be M_A unless A is open (viz Theorem 3 2 4)

A characterization for normality can also be derived in a similar manner But the condition obtained on C(X) for a space to be normal seems to be artificial

<u>Theorem</u> 1 3 A space X is normal if and only if for any two disjoint closed sets A and B there exist two countably generated ideals I and J such that $I \subseteq M_A$ and $J \subseteq M_B$ with $(M_{\bigcap Z(I)}, M_{\bigcap Z(J)}) = C(X)$

Proof X normal $\Longrightarrow$ A and B are contained in disjoint zero-sets $Z(f)$ and $Z(g)$ respectively for some $f, g \in C(X)$ By the lemma we get I and J with the desired property Conversely, let $I = (f_1, f_2, \ )$ and $J = (g_1, g_2, \ )$ It is easily seen that $\bigcap Z(I) = \bigcap_{f \in I} Z(f) = \bigcap_{i \in N} Z(f_i)$ and $\bigcap Z(J) = \bigcap_{i \in N} Z(g_i)$ Let f and $g \in C(X)$ s t $Z(f) = \bigcap_{i \in N} Z(f_i)$ and $Z(g) = \bigcap_{i \in N} Z(g_i)$ Since $(M_{\cap Z(I)}, M_{\cap Z(J)}) = C(X)$, $Z(f) \cap Z(g) = \emptyset$ as $f \in M_{\cap Z(I)}$ and $g \in M_{\cap Z(J)}$ But $Z(f) \supset A$ and $Z(g) \supset B$ and we get that A and B are contained in disjoint zero-sets #

Remark 1 2 Let T denote the Tychonoff plank We know that T is not normal and that $C(T) \cong C(\beta T)$ We see that the above characterization is not an algebraic invariant

2 Some particular countably generated ideals

The conditions in (3 1 1) lemma give rise to two questions What are those closed sets A for which O_A or M_A is countably generated ? As will be clear from the following results, the desired conditions on A for the above two assertions to be true are completely analogous to those obtained for the case when A was a point

Theorem 2 1 Let F be a closed set, completely separated from every disjoint closed set in X Then O_F is countably generated if and only if F has a countable base of neighborhoods in X

Proof $O_F = (f_1, f_2, \ )$ implies that $\{Z(f_1), Z(f_2), \ \}$ generates $Z[O_F]$

Claim Finite intersections of members of $\{Z(f_1), Z(f_2), \ \}$ constitute a countable base of neighborhoods of F

For, if U is an open set containing F, then $X-U$ will be a closed set completely separated from F. Let $f \in C(X)$ with $f(F) = 0$ and $f(U) = 1$. Let $g \in C(X)$ be such that $Z(g) = f^{-1}[0,\frac{1}{2}]$ (cf. Proposition (1.2.2(a))). Thus $Z(g)$ is a zero-set neighborhood of F and $(X-U) \cap Z(g) = \emptyset$. We have $g \in O_F$ and $Z(g) \subset U$. Since $g \in O_F$, $g = \sum_L k_i f_i$, with $k_i \in C(X)$, where L is a finite subset of N. Clearly $Z(g) \supset \bigcap_L Z(f_i)$. Thus $F \subset \bigcap_L Z(f_i) \subset Z(g) \subset U$, and the claim is true. Conversely, let $\{U_i\}_{i \in N}$ be an open base of neighborhoods for F. It is clear that $F = \bigcap_{i=1}^{\infty} U_i$. By hypothesis, for each i there exists a g_i in $C(X)$ with $g_i(F) = 0$ and $g_i(X-U_i) = 1$.

Claim. The collection $\{f_1, f_2, \ldots\}$, where $f_i \in C(X)$ and $Z(f_i) = g_i^{-1}[0, \frac{1}{2}]$, generates O_F.

For, if $f \in O_F$ then there exists some U_i with $Z(f) \supset U_i \supset F$. Hence $Z(f) \supset U_i \supset Z(f_i) \supset F$. Proposition (1.2.3(a)) implies the existence of a $k \in C(X)$ such that $f = k f_i$. #

<u>Corollary</u> 2.2. Let X be normal and F closed in X. Then O_F is countably generated if and only if F has a countable base of neighborhoods in X.

Since in a compact space a zero-set has a countable base of neighborhoods (corollary 2.4.4), we have the following.

<u>Corollary</u> 2.3. Let F be a closed set in a compact space X. Then O_F is countably generated if and only if F is a zero-set.

The following example of a space X in which A is a closed set such that O_A is countably generated but A does not have a countable base of neighborhoods was discovered by Kohls [14]

<u>Example</u> 2 1 Let $X = \beta P - (\beta Q - Q)$ Consider the ideal

$I = \{f \in C(X)$ $X-Z(f)$ is a bounded subset of $P\}$

I is countably generated

Let q be a rational Let B_q be the set of all open intervals symmetric about q with rational end points Let $A \in B_q$ Since R is perfectly normal, there exists a function $f \in C(R)$ such that $Z(f) = R - A$ Since A is bounded, the extension of f to βP will vanish on $\beta R - A$ Thus for each A in B_q we get a function $f \in C(X)$ such that $Z(f) = X-A$ If $A = (q-q', q+q')$, then denote the f obtained above, for this interval, by $f_{qq'}$ It is now clear that the set $\{f_{qq'}$ $q, q' \in Q\}$ generates I For, if $f \in I$, consider a rational $q \in X-Z(f)$ and another large enough rational q' such that $(q-q', q+q') \supset X-Z(f)$ and q' is an interior point of $Z(f)$ Thus $Z(f)$ is a neighborhood of $Z(f_{qq'})$ and Proposition (1 2 3(a)) gives the result

Let $A = X - R$ It is clear that $O_A = I$

A does not have a countable base of neighborhoods

Let $\{U_n\}$ be any sequence of neighborhoods of A Since each U_n must contain all irrationals outside some bounded set, we may choose an increasing sequence of distinct positive integers $\{k_n\}$ such that U_n contains all irrationals with absolute value exceeding k_n

For each n, select any rational $q_n \in U_n$ with $k_n < q_n < k_{n+1}$ Then $V = X - \{q_1, q_2, \quad \}$ is a neighborhood of A, but V contains no U_n Thus, A does not have a countable base of neighborhoods

In the above example we observe that A is not completely separated from Q

In [14] Kohls gives another example of a space in which a closed set A has countable base of neighborhoods but 0_A is not countably generated However, we observe that the closed set A in this example does not have a countable base of neighborhoods and thus the example does not serve the purpose

The space considered for the above mentioned example is the space Γ of Example 2 5 2 and the closed set A is the set Q Since Q is not a zero-set, Lemma (3 1 1) implies that 0_Q cannot be countably generated We prove that Q does not have a countable base of neighborhoods in Γ

Let $\{U_i\}$ be a decreasing sequence of neighborhoods of Q We choose a subsequence of $\{U_i\}$ as follows

We say $U_1 \supsetneqq U_{p_1}$ at 1 if $B_{(1,\frac{1}{2})} \cap U_1 \supsetneqq B_{(1,\frac{1}{2})} \cap U_{p_1}$ Similarly we say $U_{p_1} \supsetneqq U_{p_2}$ at 2 if $B_{(2,\frac{1}{3})} \cap U_{p_1} \supsetneqq B_{(2,\frac{1}{3})} \cap U_{p_2}$ and so on, where $B_{(r,s)}$ denotes the disc of radius s with center r We consider the sequence $\{U_{p_0=1}, U_{p_1}, U_{p_2}, \quad \}$ thus constructed Let q_j be an element in Γ such that $q_j \in B_{(j,\frac{1}{j+1})} \cap U_{p_{j-1}}$ and does not belong to $B_{(j,\frac{1}{j+1})} \cap U_{p_j}$ We get a sequence $\{q_i\}$ in Γ which is closed by

construction Thus $\Gamma - \{q_1\}$ is an open set contaning Q and not containing any U_{p_i}, and hence $\Gamma - \{q_1\}$ does not contain any U_i either

This way we are left with the follwoing

Question 2 Let A have a countable base of neighborhoods in X where A is closed in X Does it always follow that O_A is countably generated ?

The above theorem helps us in getting examples of rings in which countable intersection of countably generated ideals need not be countably generated Consider the space Γ of the above discussion

It is plain that the space Γ satisfies first axiom of countability Thus O_q, for each $q \in \Gamma$, is countably generated But $O_Q = \bigcap_{q \in Q} O_q$ is not countably generated Thus $C(\Gamma)$ is a commutative ring with identity in which countable intersection of countably generated ideals is not countably generated

Another interesting example of such a ring is $C(R)$ Infact, proposition (1 2 11) implies that Z does not have a countable base of neighborhoods in R Since R is normal, O_Z will not be countably generated But O_n for each $n \in Z$ is countably generated since R is first countable

The next theorem in this sequence was proved by Gillman [8] for compact subsets A of X It was pointed out by Kohls [14] that the arguments in that proof could be modified to get a proof of the theorem for closed subsets A of X

Theorem 2 4 Let A be closed in X Then M_A is countably generated if and only if A is open in X

Proof Let $M_A = (f_1, f_2, \quad)$ Consider $h = \sum 2^{-n} |f_n|^{1/2}$ Since $\sum 2^{-n} |f_n|^{1/2}$ is a uniformly convergent series of continuous functions, h is a continuous function Also, it is easy to verify that $Z(h) = A$ Thus $h \varepsilon M_A$ and hence $h = \sum_{i=1}^{n} g_i f_i$ for some $g_i \varepsilon C(X)$ If $a \varepsilon A$ and $m = \max_{1 \leq i \leq n} |g_i(a)|$, then $W = \bigcap_{i=1}^{n} g_i^{-1}(-m-1, m+1)$ will be an open neighbourhood of a on which each g_i will be bounded by m+1

$$h(x) \leq (1+m)(|f_1(x)| + |f_2(x)| + \quad + |f_n(x)|) \text{ for } x \varepsilon W$$

Since each f_i is zero on a, there exists a neighborhood U of a such that $|f_i(x)|^{1/2} < \frac{2^{-n}}{m+1}$ for each $1 \leq i \leq n$ and $x \varepsilon U$

$$(1+m)\ |f_i(x)| < 2^{-n}\ |f_i(x)|^{1/2} \text{ whenever } |f_i(x)| \neq 0 \text{ and } x \varepsilon U$$

Thus, if $x \varepsilon W \cap U$ and $f_i(x) \neq 0$ for any $1 \leq i \leq n$, we have

$$h(x) \leq (m+1)(|f_1(x)| + |f_2(x)| + \quad + |f_n(x)|) < 2^{-n}(|f_1(x)|^{1/2} + + |f_2(x)|^{1/2} + \quad + |f_n(x)|^{1/2}),$$ a contradiction to the definition of h

Hence $x \varepsilon W \cap U$ implies that $f_i(x) = 0$ for all $1 \leq i \leq n$ We get that $W \cap U \subset Z(h) = A$ But $W \cap U$ is a neighborhood of a This gives that a is an interior point of A and hence A is open

Conversely, if A is open and closed both then A is a zero-set Let $A = Z(f)$, for some $f \varepsilon C(X)$ Clearly f generates M_A in view of proposition (1 2 3(a) #

<u>Corollary</u> 2 5 M_A is not countably generated for any closed A properly contained in a completely regular Hausdorff connected space

The conditions on a closed set A for which O_A or M_A is finitely generated are easy to derive In fact, from the proof of the Lemma (3 1 1) it is clear that if I is an ideal generated by $\{f_\alpha \quad \alpha \in \Lambda\}$ for some indexing set Λ and if $O_A \subsetneqq I \subsetneqq M_A$ for some closed set A, then $A = \bigcap_{f_\alpha \in \Lambda} Z(f_\alpha) = \bigcap Z(O_A) = \bigcap Z(M_A)$ Thus if O_A is finitely generated, A becomes a neighborhood of itself and hence is open Conversely, if A is open then, A is a zero-set and it is clearly seen that O_A is generated by a function f such that $Z(f) = A$ We proved the following

<u>Proposition</u> 2 6 For any closed set A in X, O_A is finitely generated if and only if A is open

We next observe

<u>Theorem</u> 2 7 Let A be a closed subset of X Then, M_A is finitely generated if and only if A is open

Proof Clear from Theorem 3 2 4 #

<u>Corollary</u> 2 8 For any closed set A of X, M_A countably generated $\Longleftrightarrow$ M_A finitely generated $\Longleftrightarrow$ M_A principal $\Longleftrightarrow$ A open

§3 Countably generated O^A and M^A for A closed in βX

The necessary and sufficient conditions on a closed set A of βX for which O^A is countably generated are exactly the same as for O_A where A is closed set of X Precisely we have the following

<u>Theorem</u> 3 1 Let A be a closed set in βX Then O^A is countably generated if and only if A is a zero-set in βX

Proof Let $O^A = (f_1, f_2, \quad)$

Claim $\bigcap_{g \in O^A} \operatorname{Int} \overline{Z(g)}^{\beta X} = \bigcap_{i=1}^{\infty} \operatorname{Int} \overline{Z(f_i)}^{\beta X}$ For, if $g \in O^A$, then $g = \sum_{\text{finite}} a_i f_i$ for some $a_i \in C(X)$, so that $Z(g) \supset \bigcap_{\text{finite}} Z(f_i)$

$$\overline{Z(g)}^{\beta X} \supset \overline{\bigcap_{\text{finite}} Z(f_i)}^{\beta X} = \bigcap_{\text{finite}} \overline{Z(f_i)}^{\beta X}$$

$$\operatorname{Int} \overline{Z(g)}^{\beta X} \supset \operatorname{Int} \bigcap_{\text{finite}} \overline{Z(f_i)}^{\beta X} = \bigcap_{\text{finite}} \operatorname{Int} \overline{Z(f_i)}^{\beta X}$$

$$\bullet \quad \bigcap_{g \in O^A} \operatorname{Int} \overline{Z(g)}^{\beta X} \supset \bigcap_{i=1}^{\infty} \operatorname{Int} \overline{Z(f_i)}^{\beta X}$$ That

$\bigcap_{i=1}^{\infty} \operatorname{Int} \overline{Z(f_i)}^{\beta X} \supset \bigcap_{g \in O^A} \operatorname{Int} \overline{Z(g)}^{\beta X}$ is clear as $f_i \in O^A$ for eah $i \in N$

By proposition (1 2 9), $A = \bigcap_{i=1}^{\infty} \operatorname{Int} \overline{Z(f_i)}^{\beta X}$ Thus A is a G_δ in βX Since βX is normal, proposition (1 2 1) implies that A is a zero-set

Conversely, let A be a zero-set Corollary (3 2 3) implies that $O_A^{\beta X}$ is countably generated in $C(\beta X)$ Let $O_A^{\beta X} = (f_1, f_2, \quad)$, where $f_i \in C(\beta X)$ Also, let $g \in O^A$ By proposition (1 2 3(a)) there exists a unit $u \in C(X)$ such that $u\,g = (-1 \vee g) \wedge 1 = \underline{g}$ (say) Thus $Z_X(\underline{g}) = Z_X(g)$ and $\underline{g}$ is bounded Let $\tilde{\underline{g}}$ be the extension of $\underline{g}$ to βX We have $Z_{\beta X}(\tilde{\underline{g}}) \supset \overline{Z_X(\underline{g})}^{\beta X} = \overline{Z_X(g)}^{\beta X}$ As $g \in O^A$, $\overline{Z_X(g)}^{\beta X}$ is a neighborhood of A So $Z_{\beta X}(\tilde{\underline{g}})$ is also a neighborhood of A and hence

$\tilde{\underline{g}} \in O_A^{\beta X}$ We get that $\tilde{\underline{g}} = \sum_{\text{finite}} a_i f_i$ for some $a_i \in C(\beta X)$

$$\tilde{\underline{g}}/X = \underline{g} = \sum_{\text{finite}} a_i f_i/X = \sum a_i/X \quad f_i/X$$

From this it becomes clear that $(f_1/X, f_2/X, \quad)$ generates O^A #

The following corollary is trivial

<u>Corollary</u> 3 2 O^A is countably generated if and only if $O^A_{\beta X}$ is countably generated for any closed set A in βX

<u>Remark</u> 3 1 The proof of the converse of the above theorem can easily be modified to prove the following

<u>Proposition</u> 3 3 $M_A^{\beta X}$ countably generated $\Longrightarrow M^A$ countably generated for any closed set A in βX

Also, from the proof of the theorem (3 3 1) we can extract the arguments to prove

<u>Proposition</u> 3 4 Let A be closed in βX

These are equivalent

(1) O^A is finitely generated

(2) $O_A^{\beta X}$ is finitely generated

(3) A is open in βX

(2) $\Longleftrightarrow$ (3) has been proved in proposition (3 2 6)

Let A be an open as well as closed set in βX Corollary (3 2 7) implies that $M_A^{\beta X}$ is principal and hence M^A is countably generated by proposition (3 3 3) The converse, i e , M^A countably generated implies A open for any closed A in βX was left as an open problem in [5]

We settle the problem for pseudocompact spaces We also give a counter example to show that in a general space the above mentioned converse is not true In fact, it was observed later that, G De Marco [4] had also pointed out a counter example for the same purpose and his example resembles the one given here excepting that we do not use the concept of round filters

To prove the final result of this section We require the following lemma

<u>Lemma</u> 3 5 Let X be pseudocompact and $g \in C(\beta X)$ Also let $\underline{g} = g|X$ Then, $Z_{\beta X}(g) = \overline{Z_X(\underline{g})}^{\beta X}$

Proof It is plain that $Z_{\beta X}(g) \supseteq \overline{Z_X(\underline{g})}^{\beta X}$, and also that $Z_X(\underline{g}) = Z_{\beta X}(g) \cap X \neq \emptyset$ (proposition (1 2 8(c))) Next, let $Z_{\beta X}(g) = \bigcap\limits_{n=1}^{\infty} \overline{Z_X(f_n)}^{\beta X}$, where $f_n \in C(X)$, $n \in N$, then

$Z_{\beta X}(g) \supset \bigcap\limits_{n=1}^{\infty} Z_X(f_n)$

$$\therefore \; Z_{\beta X}(g) \cap X \supset \bigcap_n Z_X(f_n)$$

$$\begin{aligned}
\overline{Z_{\beta X}(g) \cap X}^{\beta X} &\supset \overline{\bigcap_n Z_X(f_n)}^{\beta X} \\
&= \overline{\bigcap_n Z_X(f_n)}^{\upsilon X} \\
&= \bigcap_n \overline{Z_X(f_n)}^{\upsilon X} \\
&= \bigcap_n \overline{Z_X(f_n)}^{\beta X} \\
&= Z_{\beta X}(g)
\end{aligned}$$

But $Z_{\beta X}(g) \cap X = Z_X(\underline{g})$

$$\overline{Z_X(\underline{g})}^{\beta X} \supseteq Z_{\beta X}(g)$$

In the above proof we have used the proposition (1 2 8(d))

Proposition 3 6 Let A be any subset of a pseudocompact space X Then, M^A countably generated $\Longrightarrow$ $M_A^{\beta X}$ is countably generated

Proof Let $M^A = (f_1, f_2, \quad)$ and $g \varepsilon M_A^{\beta X}$ By above lemma, $Z_{\beta X}(g) = \overline{Z_X(\underline{g})}^{\beta X}$, where $\underline{g} = g|X$ Since $g \varepsilon M_A^{\beta X}$, $Z_{\beta X}(g) \supset A$, and hence $\overline{Z_X(\underline{g})}^{\beta X} \supset A$ giving us $\underline{g} \varepsilon M^A$ There exist finitely many functions $f_1, f_2, \quad , f_n$ (say) such that $\underline{g} = \sum_{i=1}^{n} a_i f_i$ where $a_i \varepsilon C(X)$ Since X is pseudocompact, each of a_i and f_i can be extended to βX If $\tilde{f}$ denotes the extension of a function $f \varepsilon C(X)$ to βX, then $g = \tilde{\underline{g}} = \sum_{i=1}^{n} \tilde{a}_i \tilde{f}_i$, because of unique extension property preserved by βX We get that$(\tilde{f}_1, \tilde{f}_2, \quad) = M_A^{\beta X}$ #

The propositions (3 3 3) and (3 3 6) are combined to give the following theorems

Theorem 3 7 Let A be a closed subset of βX The following are equivalent for a pseudocompact space X

(1) M^A is countably generated

(2) $M_A^{\beta X}$ is countably generated

(3) A is open in βX

As was mentioned earlier, the above theorem is not true in a general space

Example 3 1 Take a non discrete P-space X From proposition (1 2 13(b)) it is clear that X is not pseudocompact Thus proposition (1 2 8(c)) ensures the existence of a zero-set A in βX which does not intersect X But proposition (1 2 13(a)) implies $O^A = M^A$ Since $A \cap X = \emptyset$, A cannot be open in βX By Theorem (3 3 1) O^A is countably generated Thus M^A is countably generated but A is not open in βX

As an application of theorem 3 1, we will prove the following well known theorem of [9]

Theorem 3 8 Every non-empty zero-set in βX, if disjoint from X, has at least 2^c elements

To prove this theorem, we will make use of the following theorem of Gillman [8]

"Each free, countably generated ideal is contained in 2^c hyper-real maximal ideals but in no real maximal ideal"

Proof of Theorem 3 8 Let A be a zero-set in βX with $A \cap X = \emptyset$ Theorem 3 1 implies that O^A is countably generated It is clear that O^A is free The above theorem of Gillman applies to give the result #

Since Z does not have a countable base of neighborhoods in R (Proposition 1 2 11), C(R) is an example of a ring in which O_A need not be countably generated for each closed set A in R But if X is

a compact perfectly normal space, then corollary 3 2 3 implies that O_A is countably generated in $C(X)$ for every closed set A in X Thus, the class of spaces X whose $C(X)$ satisfy the above condition lies between the class of prefectly normal spaces and the class of compact perfectly normal spaces As is shown by $\mathbb{R}$, metric spaces need not fall in this class It will be interesting to identify the above mentioned class

CHAPTER 4

STUDY VIA RING ISOMORPHISM

In [1] Anderson proved that if X and Y are both completely regular wpn spaces, an isomorphism between their rings of continuous functions implies a homeomorphism between X and Y. We prove that if X is a completely regular, wpn, realcompact space, then the isomorphism $h: C(X) \to C(Y)$ implies X and Y are homeomorphic. This differs from similar theorems in the literature in that we do not put any condition on Y except that it is completely regular. We prove the main theorem through two propositions about images of M_p and O_p under an isomorphism. A similar proof works for Anderson's theorem also.

<u>Proposition</u> 1.1. Let X and Y be two spaces with h an isomorphism between $C(X)$ and $C(Y)$. If $p \in X$, then $h(M^p) = M^q$ for some $q \in Y$. Also, then $h(O^p) = O^q$.

Proof. Since under an onto homomorphism a maximal ideal goes to a maximal ideal, the first assertion is clear. Since Theorem (1.2.4) gives us that a z-ideal is intersection of prime ideals, O^p is intersection of all prime ideals between O^p and M^p. It is clear that $h(O^p)$ is the intersection of all prime ideals between O^q and M^q, since image of a prime ideal is a prime ideal under a one-one homomorphism. This shows that $h(O^p) = O^q$.

We use the above proposition to give another proof of Lemma 3.3.5.

Let A be a zero-set in X There exists a principal ideal I in $C(X)$ with $O_A \subseteq I \subseteq M_A$ Since X is pseudocompact, let h be the isomorphism between $C(X)$ and $C(\beta X)$ such that $h(f) = f^\beta$ where f^β is the extension of f to βX Thus for $p \in X$, $h(M_p) = \{f^\beta | f \in M_p\}$ Because of the unique extension of $f \in C(X)$ to βX, it is clear that $h(M_p) = M_p^{\beta X}$ From the above proposition, we get that

$h(O_p) = O_p^{\beta X}$

$$h(O_A) = \bigcap_{a \in A} O_a^{\beta X} \text{ and } h(M_A) = \bigcap_{a \in A} M_a^{\beta X}$$

$$O_A^{\beta X} \subseteq h(I) \subseteq M_A^{\beta X}$$

$$O_{\bar{A}^{\beta X}}^{\beta X} \subseteq O_A^{\beta X} \subseteq h(I) \subseteq M_A^{\beta X} = M_{\bar{A}^{\beta X}}^{\beta X}$$

Since $h(I)$ is principal and $\bar{A}^{\beta X}$ is closed in βX, $\bar{A}^{\beta X}$ is a zero-set in βX #

<u>Corollary</u> 1 2 Let X be a pseudocompact space A point zero-set in X is a zero-set in βX

Proof In the above proposition take a point zero-set in place of the zero-set A #

<u>Corollary</u> 1 3 A pseudocompact wpn space is first countable

Proof The proof follows from Corollaries 2 1 4 and 4 4 2 #

The above corollary proves (5) of Theorem 2 4 8

<u>Corollary</u> 1 4 Every G_δ-point in X is a G_δ-point in υX

Proof In the above proposition take $C(\upsilon X)$ in place of $C(\beta X)$ and define h as $h(f) = f^{\upsilon}$, the extension of $f \in C(X)$ to υX #

Before proving the main theorem, we will prove the following

Proposition 1 5 Let X and Y be two spaces such that $h : C(X) \to C(Y)$ is an isomorphism & $p \varepsilon X$ be a zero-set in X Then $h(M_p)$ is also a fixed maximal ideal

Proof Since $\frac{C(X)}{M_I}$ is isomorphic to $\frac{C(Y)}{h(M_p)}$, $h(M_p)$ is a real maximal ideal If $h(M_p) = M^{p'}$ for some $p' \varepsilon Y$, then by proposition 4 1 4, $h(O_p) = O^{p'}$ Also, if I is a countably generated ideal between O_p and M_p, then h(I) is also countably generated, since h is an isomorphism If $\{f_i : i \varepsilon N\}$ generates I, then $\bigcap Z[h(I)] = \bigcap_{i=1}^{\infty} Z[h(f_i)] = p'$ or $\emptyset$ Since $M^{p'}$ is real, $Z[M^{p'}]$ is closed under countable intersection Thus $\bigcap Z[h(I)] = p'$, and we get $p' \varepsilon Z[M^{p'}]$, hence $Z[M^{p'}]$ is fixed #

Theorem 1 6 Let X be wpn and realcompact Let Y be another space such that $C(X) \cong C(Y)$ Then X and Y are homeomorphic

Proof Let h be an isomorphism between X and Y The above proposition implies that for $p \varepsilon X$, $h(M_p) = M_{p'}$ for some $p' \varepsilon Y$ Define $f : X \to Y$ such that $f(p) = p'$

Let $M_{p'}$ be some maximal ideal in C(Y) It is clear that $h^{-1}(M_{p'})$ will be real Since X is realcompact, $h^{-1}(M_{p'})$ is fixed. We get that f is onto Next, $f(p_1) = f(p_2) \implies h^{-1}(M_{f(p_1)}) = h^{-1}(M_{f(p_2)})$

$$\implies M_{p_1} = M_{p_2} \implies p_1 = p_2$$

Thus f is one-one also

It is easy to observe that $h(M_A) = M_{f(A)}$ Infact, $h(M_A) = h(\bigcap_{a\in A} M_a) = \bigcap_{a\in A} h(M_a) = \bigcap_{a\in A} M_{f(a)} = M_{f(A)}$ Similarly by proposition (4 1 1) we have $h(O_A) = O_{f(A)}$

Claim f is a closed map

We can prove even a stronger statement that f takes zero-sets onto zero-sets

For, if A is a zero-set, there exists a countably generated ideal I such that $O_A \subsetneq I \subsetneq M_A$ Hence $h(O_A) \subseteq h(I) \subseteq h(M_A)$ and we get $O_{f(A)} \subsetneq h(I) \subsetneq M_{f(A)}$ Since $h(I)$ is countably generated, the claim will be true if we show that $f(A)$ is closed

Let $f(A)$ be not closed Clearly $M_{f(A)} = M_{\overline{f(A)}}$ But $M_{f(A)} = h(M_A)$

$$h^{-1}(M_{\overline{f(A)}}) = M_A$$

$$M_{f^{-1}\overline{(f(A))}} = M_A$$

$$f^{-1}\overline{(f(A))} \subset A, \text{ since } A \text{ is closed}$$

$$\overline{f(A)} \subset f(A), \text{ and we get that } f(A) \text{ is closed}$$

Similarly it can be proved that f^{-1} is closed

Remark 1 1 In the above Theorem, the realcompactness of X was used in proving that f is onto #

Remark 1 2 If in the above theorem we have Y also to be wpn, then applying Proposition(4 1 5) for h^{-1}, we see that $h^{-1}(M_p)$ is fixed for

any $p \in Y$ We get that the function h defined in the proof of above theorem is onto Hence the above proof can be refined to get

Theorem 1 7 (Anderson) Let X and Y be wpn such that $C(X) \cong C(Y)$ Then, X is homeomorphic to Y #

Hewitt [11] proved the following theorem

Theorem 1 8 The ring $C(X)$ completely determines X if and only if X is realcompact with the property that for every non-isolated point $p \in X$, there is a function in $C(X-\{p\})$ which cannot be continuously extended over X, i e , $X - \{p\}$ is not C-embedded in X

From Theorem 1 6 we have that if X is wpn and realcompact, then $C(X)$ completely determines X Comparing this with Hewitt's theorem, we at once get

Corollary 1 9 Let X be realcompact, wpn space Then, $X - \{p\}$ is not C-embedded in X for any nonisolated point $p \in X$

Since N is C*-embedded in the space Σ (cf Example 1 1 4) and Σ is wpn, realcompact (cf Theorem 1 2 10), the term "C-embedded" in Corollary(4 1 9)cannot be replaced by "C*-embedded"

§2 Some subrings of $C(X)$

Let X be a space and F a subset of X Define $R_F = \{f \in C(X)$ such that $f(F)$ is a constant$\}$ It is clear that R_F is closed under the operations + and , and that R_F is a subring of the ring $C(X)$ Also, 1 belongs to R_F.

Theorem 2 1 $C(X/F) \cong R_F$

Proof Let τ $X \to X/F$ be the quotient map If f $X/F \to R$ is any element in $C(X/F)$, define $h(f) = f \circ \tau$ Clearly h is a well defined map from $C(X/F)$ to R_F Next, define $\underline{g}$ $X/F \to R$ such that $\underline{g} \circ \tau = g$ for any $g \in R_F$ Since X/F has the quotient topology, $\underline{g}$ is continuous Thus h is onto h is clearly one-one We prove that h is a homomorphism

$$\begin{aligned} h(f\ g)\ (x) &= ((f\ g) \circ \tau)(x) = (f\ g)\ (\tau(x)) \\ &= f(\tau(x)) \quad g(\tau(x)) \\ &= f \circ \tau(x) \quad g \circ \tau(x) \\ &= (h(f) \quad h(g))(x) \end{aligned}$$

Similarly, $h(f + g) = h(f) + h(g)$

Hence h is an isomorphism #

Since we are interested in studying $C(X)$ for a completely regular space X, we will require X/F to be completely regular If X is completely regular and if F is a closed subset of X, completely separated from any disjoint closed set, it is easily seen that X/F is completely regular Also, if X is normal and F is closed in X, then X/F is normal

Corollary 2 2 A closed G_δ in a normal space X is a zero-set

Proof Let F be a closed G_δ in X It is clear that in X/F, the point determined by F is a G_δ, hence this point is a zero-set in X/F With the help of τ we can construct a function in $C(X)$ precisely vanishing on F #

CHAPTER 5

COUNTABLY GENERATED PRIME IDEALS AND z-IDEALS

In the last chapter we observed that in a realcompact wpn space, for a non-isolated point p, $X - \{p\}$ can be C^*-embedded in X Hewitt [11] proved that in a first countable space, $X - \{p\}$ cannot be C^*-embedded in X, for any non-isolated point $p \in X$ We prove this result below and apply it to a result of Kohls [14] to get that in a first countable space a nonmaximal prime ideal cannot be countably generated Next, we prove a theorem regarding nonexistence of countably generated z-ideals above O_p, when p has a countable base of neighborhoods The proof has been deduced from a more general theorem of Kohls [14] The theorem has been generalized for some particular closed sets By the help of this theorem we identify all the countably generated z-ideals in a pseudocompact space In [4] De Marco asks whether a countably generated z-ideal I is always of the form $\bigcap_{a\in A} O^a$ with A a zero-set of βX? We prove that if it is so, even for closed sets A in βX, then $A = \theta(I)$

§1 Countably generated prime ideals

We will first prove Hewitt's result

<u>Theorem</u> 1 1 Let $p \in X$ have a countable base of neighborhoods Then $X - \{p\}$ is not C^*-embedded in X

Proof Let $\{U_n\}_{n\in N}$ be a base of neighborhoods for p, which without loss of generality, can be considered to be a decreasing sequence

Let p_1 be any point in U_1 such that it has a neighborhood V_1 contained in U_1 and such that $p \notin \bar{V}_1$. Choose a neighborhood U_{n_1} of p which does not intersect $\bar{V}_1$. Next, we choose a point $p_{n_1} \in U_{n_1}$ different from p and repeat the same process to obtain a neighborhood U_{p_2} of p and V_{n_1} ($\subseteq U_{n_1}$) of p_{n_1} such that $U_{p_2} \cap \bar{V}_{n_1} = \emptyset$. Continuing this way we will get a sequence of points $\{p_{n_0=1}, p_{n_1}, p_{n_2}, \ldots\}$ and neighborhoods $\{V_1, V_{n_1}, V_{n_2}, \ldots\}$, where each V_{n_i} is a neighborhood of p_{n_i}, with the following properties

(1) $V_{n_i} \cap V_{n_j} = \emptyset$ for $i \neq j$, and

(2) For any n, all but a finite number of the neighborhoods V_{n_k} are contained in U_n.

Define a sequence $\{\psi_{n_k}\}_{k\in N}$ of continuous functions on X such that $\psi_{n_k}(p_{n_k}) = 1$ and $\psi_{n_k}(V^C_{n_k}) = 0$, whenever k is an even integer and $\psi_{n_k} = 0$ on X, whenever k is odd. Define a function ψ on $X - \{p\}$ as follows

$$\psi(q) = \sum_{k=0}^{\infty} \psi_{n_k}(q) \quad , \; q \in X - \{p\}$$

For any $q \in X - \{p\}$, there is a neighborhood $W(q)$ of q and $U_n(p)$ of p such that $W(q) \cap U_n(p) = \emptyset$. Thus due to the property (2) of the sequence $\{V_{n_i}\}_{i\in N}$, only a finite number of the sets V_{n_k} will intersect the set $W(q)$. We get that only finitely many of ψ_{n_k} will be different from 0 on $W(q)$, and hence ψ is continuous at q, for any $q \in X - \{p\}$. Thus, ψ is continuous on $X - \{p\}$. If necessary, we form the bounded function $\bar{\psi} = \psi \wedge 1$, which is continuous on whole

of $X - \{p\}$ and see that $\bar{\psi}$ cannot be defined at p so as to be continuous at that point, since in every neighborhood $U_n(r)$, there are points $q_{n_{2k+1}}$ at which $\bar{\psi}$ vanishes and points $q_{n_{2k}}$ at which it is equal to 1 #

In [14] Kohls proved the following theorem

<u>Theorem</u> 1 2 A nonmaximal prime ideal I in $C(X)$, containing O_p, is countably generated if and only if the following three conditions are satisfied

(1) I contains an element f such that $Z(f) = \{p\}$

(2) In the class of prime z-ideals contained properly in M_p, there is exactly one maximal element Q

(3) The ideal I/Q in the ring C/Q has a countable cofinal subset #

Since the prime ideal I is nonmaximal and contains O_p, we get that p is nonisolated zero-set in X Theorem 2 6 4 gives that the statement (2) in above theorem is equivalent to the following

(2') $X - \{p\}$ is C^*-embedded in X

Theorem (5 1 1) atonce gives us

<u>Corollary</u> 1 3 Let X be a first countable space A prime ideal I in $C(X)$ is countably generated only if it is maximal

Theorem 3 2 4 gives us the following

<u>Corollary</u> 1 4 Let p be a non-isolated point having a countable base of neighborhoods If I is a prime ideal containing O_p, then I cannot be countably generated #

<u>Corollary</u> 1.5. A first countable F-space is discrete. #

§2. Countably generated z-ideals

<u>Theorem</u> 2.1. Let p have countable base of neighborhoods and I be a z-ideal above O_p. Then I is countably generated if and only if $I = O_p$.

Proof. Let I be countably generated and $I \supsetneq O_p$. Thus there exists $Z \in Z[I]$ which is not a neighborhood of p, i.e. p is not an interior point of Z. Since p has a countable base of neighborhoods, there exists a sequence $S \subset X-Z$ that converges to p. Let $I = (f_1, f_2, \ldots)$ with $|f_i| \leq 1$, $\forall i \in N$. Define a function to be zero on $\bar{Z}^{\beta X}$ and $\sum_m 2^{-m} |f_m|^{1/2}$ on S. This function is continuous on the closed set $S \cup \bar{Z}^{\beta X}$, and hence has a continuous extension to βX. Let the restriction of this function to X be denoted by h. Then $Z(h) \supset Z$, so $h \in I$ as I is a z-ideal. Let $q \in X - \{p\}$. There exists a function s in C(X) such that $s(q) = 1$ and s vanishes on a neighborhood of p, i.e. $s \in O_p$. Thus $\sum_m 2^{-m} |f_m|^{1/2}$ cannot vanish on $q (\neq p)$, hence $Z(h)$ is disjoint from S. If we somehow prove that $Z(h)$ is a neighborhood of p, we will get a contradiction as p is the limit point of S.

Since $h \in I$, $h = \sum_{\text{finite}} a_i f_i = \sum_{i=1}^{n} a_i f_i$ (say), where $a_i \in C(X)$. Let $m = \max_i |a_i(p)|$. Then $W = \bigcap_{i=1}^{n} a_i^{-1}(-m-1, m+1)$ is an open neighborhood of p on which each a_i is bounded by m+1.

$$h(x) \leq (1+m)\,(|f_1(x)| + \ldots + |f_n(x)|) \text{ for } x \in W.$$

Since each f_i is zero on p, there exists a neighborhood U of p such that $|f_i(x)|^{1/2} < \frac{2^{-n}}{m+1}$ for each $1 \leq i \leq n$ and $x \in U$.

$(1+m)\ |f_i(x)| < 2^{-n}\ |f_i(x)|^{1/2}$ whenever $|f_i(x)| \neq 0$ and $x \in U$. Thus, if $x \in W \cap U$ and if $f_i(x) \neq 0$ for some i such that $1 \leq i \leq n$, we have $h(x) < 2^{-n}(|f_1(x)|^{1/2} + \dots + |f_n(x)|^{1/2})$, a contradiction to the definition of h. Hence if $x \in W \cap U$, $f_i(x) = 0$ for all i with $1 \leq i \leq n$. We get that $W \cap U \subset Z(h)$, and that a is an interior point of $Z(h)$. #

Remark after proposition (1.2.7) shows that an ideal I in $C(X)$ need not contain $O_{\phi(I)}$. We put this condition on the ideal I to get the following theorem.

Theorem 2.2. Let I be a z-ideal in $C(X)$ with the following properties

(1) $M_{\phi(I)} \supsetneq I \supseteq O_{\phi(I)}$

(2) $\phi(I)$ is completely separated from every disjoint closed set

(3) $\phi(I)$ has a countable base of neighborhoods in X.

Then I is countably generated if and only if $I = O_{\phi(I)}$.

Proof. Due to condition (2), we have $X/\phi(I)$ a completely regular space. Let $\widetilde{\phi(I)}$ denote the point determined by $\phi(I)$ in $X/\phi(I)$. Due to condition (3), the point $\widetilde{\phi(I)}$ will have a countable base of neighborhoods in $X/\phi(I)$. Let h be the isomorphism between $C(X/\phi(I))$ and $R_{\phi(I)}$ defined in Theorem (4.2.1). It is easy to observe that $h(O_{\widetilde{\phi(I)}}) = O_{\phi(I)}$ and $h(M_{\widetilde{\phi(I)}}) = M_{\phi(I)}$. Also, from Theorem (1.2.5) we get that $h^{-1}(I)$ is a z-ideal. We have transferred the whole situation to $C(X/\phi(I))$. Applying Theorem (5.2.1) to the ring $C(X/\phi(I))$ we get that $h^{-1}(I)$ is countably generated if and only if $h^{-1}(I) = O_{\widetilde{\phi(I)}}$. The Theorem is clear. #

In a first countable space it is simple to prove that any countable set, which is also compact, has countable base of neighborhoods. Since a metric space is normal, we have

Corollary 2.3. Let X be a metric space and F a countable, compact subset of X. Then no z-ideal, excepting O_F, between O_F and M_F is countably generated.

The proof of the following corollary is also clear.

Corollary 2.4. A countably generated z-ideal I in a compact space X is of the form O_A, where A $(= \phi(I))$ is a zero-set in X.

We prove similar result for a pseudocompact space also. This is done by transferring the information obtained in $C(X)$ to $C(\beta X)$. We first give the following definitions.

Definition 2.1. Let I be an ideal in $C(X)$. Then,

(a) $I^* = I \cap C^*(X)$, where $C^*(X)$ is the ring of all bounded continuous functions on X, and

(b) $I^\beta = \{f^\beta \mid f \in I \cap C^*\}$, where f^β is the extension of f to βX.

We next prove two propositions.

Proposition 2.5: Let I be a z-ideal in $C(X)$. Then $I^* = I \cap C^*$ is a z-ideal in $C^*(X)$ and $I^\beta = \{f^\beta \mid f \in I \cap C^*\}$ is a z-ideal in $C(\beta X)$.

Proof. It is easy to verify that I^* and I^β are ideals in C^* and $C(\beta X)$ respectively. Let $f \in C^* \ni Z(f) \supseteq Z(g)$ for some $g \in I^*$. Since $C^* \subseteq C$, $f \in I$ i.e. $f \in I^*$. Next, let $f \in C(\beta X) \ni Z_{\beta X}(f) \supseteq Z_{\beta X}(g^\beta)$ for some $g^\beta \in I^\beta$. We have $Z_{\beta X}(f) \cap X \supseteq Z_{\beta X}(g^\beta) \cap X = Z_X(g)$ where

$g = g^{\beta}/X \in I^*$ Thus, $Z_X(f/X) \supset Z_X(g)$ Hence $f/X \in I^*$, i e , $f^{\beta} \in I^{\beta}$ because of unique extension of functions #

<u>Proposition</u> 2 6 Let I be an ideal such that $I \supsetneq O^{\theta(I)}$ Then I^{β} is an ideal with $I^{\beta} \supsetneq O^{\beta X}_{\theta(I)}$

Proof There exists a prime ideal P in $C(X)$ such that $P \supset O^{\theta(I)}$ and $P \not\supset I$ So there exists a bounded function $f \in I$ such that $f \notin P$, because from proposition (1 2 3(b)) we know that every ideal in $C(X)$ has a set of bounded generators Thus $P \cap C^* \not\supset I^*$ But $P \cap C^*$ is prime in C^* Since the mapping $f \to f^{\beta}$ is an isomorphism of $C^*(X)$ onto $C(\beta X)$, $(P \cap C^*)^{\beta}$ is a prime ideal not containing I^{β} We get that $I^{\beta} \neq O_{\theta(I)}$ That $I^{\beta} \supset O^{\beta X}_{\theta(I)}$ is clear #

Since every ideal in $C(X)$ has a set of bounded generators (viz , proposition 1 2 3(b)), if I is countably generated, we can have a countable set of bounded functions $(f_1, f_2, \quad)$ which generates I But $I^*(= I \cap C^*)$ need not be countably generated Though the condition seems to be very stringent , if we assume that I & I^* are countably generated, then for such ideals we have the following

<u>Theorem</u> 2 7 Let I be a countably generated ideal in $C(X)$ such that I^* is also countably generated Then I is a z-ideal only if $I = O^{\theta(I)}$

Proof Let I be a z-ideal If $I \neq O^{\theta(I)}$ then Proposition 5 2 6 implies $I^{\beta} \supsetneq O^{\beta X}_{\theta(I)}$ But since I^* is countably generated, I^{β} is also countably generated Proposition (5 2 10) (proved later) implies that $I^{\beta} \subseteq M^{\beta X}_{\theta(I)}$ Thus by Corollary (5 2 4) we get that $I^{\beta} = O^{\beta X}_{\theta(I)}$, a contradiction #

Since for a pseudocompact space X, $C(X) = C^*(X)$, I coincides with I* Thus we have the following

Corollary 2 8 Let X be a pseudocompact space and I a z-ideal in $C(X)$ Then I is countably generated only if $I = O^{\theta(I)}$ #

G De Marco [4] raised the following question

"Is every countably generated z-ideal of C(X) of the form $\bigcap_{p\varepsilon A} O^p$, with A a zero-set of βX" ?

We prove that if I is any ideal equal to $\bigcap_{p\varepsilon A} O^p$, for some closed set A in βX, then $A = \theta(I)$ Before proving this, we have two propositions

Proposition 2 9 $O^A \subseteq M^p \Longleftrightarrow p \varepsilon A$ where A is closed in βX

Proof Let $O^A \subseteq M^p$ and $p \notin A$ Then there exists $\tilde{f} \varepsilon C(\beta X)$ such that $Z_{\beta X}(\tilde{f}) \supseteq A$ and $\tilde{f}(p) = 1$ Consider $\tilde{g} \varepsilon C(\beta X)$ such that $Z_{\beta X}(\tilde{g}) = \tilde{f}^{-1}[0,\frac{1}{2}]$ Thus $Z_{\beta X}(\tilde{g})$ is a neighborhood of A and $p \notin Z_{\beta X}(\tilde{g})$ Let $g = \tilde{g}/X$ Then $Z(g) = Z_{\beta X}(\tilde{g}) \cap X \supseteq \text{Int } Z_{\beta X}(\tilde{g}) \cap X$ Thus $\overline{Z(g)}^{\beta X} \supseteq \overline{\text{Int } Z_{\beta X}(\tilde{g}) \cap X}^{\beta X} = \overline{\text{Int } Z_{\beta X}(\tilde{g})}^{\beta X}$, as X is dense in βX Hence $\overline{Z(g)}^{\beta X}$ is a neighborhood of A in βX We get that $g \varepsilon O^A$ Since $Z_{\beta X}(\tilde{g}) \supseteq \overline{Z(g)}^{\beta X}$, $p \notin \overline{Z(g)}^{\beta X}$ This implies that $g \notin M^p$, a contradiction to the fact that $O^A \subseteq M^p$ Thus $p \varepsilon A$

Converse follows by observing that $O^A = \bigcap_{p\varepsilon A} O^p$ #

Proposition 2 10

$$\bigcap_{p\varepsilon\theta(I)} O^p \subseteq I \subseteq \bigcap_{p\varepsilon\theta(I)} M^p$$

for any ideal I

Proof $\theta(I) = \bigcap_{f\varepsilon I} \overline{Z(f)}^{\beta X} = \{p \ \varepsilon \ \beta X \text{ such that } I \subseteq M^p\}$ #

Thus it is clear that $I \subseteq \bigcap_{p\varepsilon\theta(I)} M^p$ Next, let $f \varepsilon \bigcap_{p\varepsilon\theta(I)} O^p$

Then $\mathrm{Int}_{\beta X} \overline{Z(f)}^{\beta X} \supset \theta(I)$ $X - \mathrm{Int}_{\beta X} \overline{Z(f)}^{\beta X}$ is a closed set in βX

Since the family $\{\overline{Z(f)}^{\beta X}$ such that $f \varepsilon I\}$ has finite intersection property, there exists an $f_1 \varepsilon I$ such that $\overline{(f_1)}^{\beta X} \cap (X - \mathrm{Int}_{\beta X} \overline{Z(f)}^{\beta X}) = \emptyset$, as βX is compact and $\theta(I) = \bigcap_{f\varepsilon I} \overline{Z(f)}^{\beta X}$ Thus $\theta(I) \subset \overline{Z(f_1)}^{\beta X} \subset \mathrm{Int}_{\beta X} \overline{Z(f)}^{\beta X}$ Hence Proposition (1 2 3(a)) implies that $f = k\, f_1$ for some $k \varepsilon C(X)$ Since $f_1 \varepsilon I$, $f \varepsilon I$ #

<u>Theorem</u> 2 11 Let $I = \bigcap_{p\varepsilon A} O^p$, for some closed set A in βX Then $A = \theta(I)$

Proof From Proposition (5 2 10) we have

$$\bigcap_{p\varepsilon\theta(I)} O^p \subseteq I \subseteq \bigcap_{p\varepsilon A} O^p \subseteq \bigcap_{p\varepsilon\theta(I)} M^p$$

From Proposition (5 2 9) it is clear that $\theta(I) \subseteq A$ Conversely, if $a \varepsilon A$, then

$$\bigcap_{p\varepsilon\theta(I)} O^r \supseteq \bigcap_{p\varepsilon\theta(I)} O^p \cap O^a$$

But since $\bigcap_{p\varepsilon\theta(I)} O^p \subseteq \bigcap_{p\varepsilon A} O^p$ and $a \varepsilon A$,

$$\bigcap_{p\varepsilon\theta(I)} O^p \subseteq \bigcap_{p\varepsilon\theta(I)} O^p \cap O^a$$

$$\bigcap_{p\varepsilon\theta(I)} O^p = \bigcap_{p\varepsilon\theta(I)} O^p \cap O^a$$

$$\bigcap_{p\varepsilon\theta(I)} O^p \subseteq O^a \subseteq M^a$$

Hence again by Proposition (5 2 9) $a \varepsilon \theta(I)$

$$A \subseteq \theta(I) \qquad \#$$

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